Using Intuitive Geometry - Exercise 9

Due Date: November 3rd - Instructor: Felix Breuer

Announcement

This is the final exercise!

Exercises

1) For any lattice point $x \in \mathbb{Z}^d$ we write z^x to denote the monomial $z_1^{x_1} \cdot \ldots \cdot z_d^{x_d}$. We identify finite sets of lattice points $A \subset \mathbb{Z}^d$ with Laurent polynomials with coefficients in $\{0, 1\}$ via

$$\phi : A \mapsto \sum_{x \in A} z^x.$$

Note that $\phi(A)(z) = \phi(A)(z_1, \ldots, z_d)$ is a polynomial in d variables. The point of this problem is to study what happens geometrically when we substitute monomials z^{v_i} for each of the variables z_i.

Let $v_1, \ldots, v_d \in \mathbb{Z}^d$ be lattice points. Consider the polynomial

$$\phi(A)(z^{v_1}, \ldots, z^{v_d})$$

and let

$$B := \phi^{-1}(\phi(A)(z^{v_1}, \ldots, z^{v_d}))$$

denote the corresponding set of lattice points.

Give a geometric (linear algebra) description of B in terms of A and $v_1, \ldots, v_d$.

2) Consider the rational polytope $P = [-\frac{2}{3}, \frac{1}{2}]$. Embed $P \times \{1\}$ in $\mathbb{R}^2$. Consider the cone C generated by the vertices of $P \times \{1\}$, i.e., let

$$C = \text{cone}\left((-\frac{2}{3}, 1),\ (\frac{1}{2}, 1) \right).$$

1. Draw C and the lattice $\mathbb{Z}^2$ (in a suitable region around the origin).
2. Enumerate the set of lattice points in the fundamental parallelepiped of C.
3. Write the multivariate generating function $\sum_{x \in C \cap \mathbb{Z}^2} z^x$ as a rational function.
4. Write the (univariate) Ehrhart series $\sum_{k \geq 0} L_P(k) z^k$ of P as a rational function.

Questions?

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