Using Intuitive Geometry - Exercise 8

Due Date: October 25th (!!!) - Instructor: Felix Breuer

Announcement

In total there will be 9 exercises! For exercises 8 and 9 you will have $1\frac{1}{2}$ weeks each. Exercise 8 (this one) is due October 25th and exercise 9 will be due Nov 3rd (first day of seminars).

Exercises

The point of this exercise is to do the Beck-Zaslavsky proof of Stanley's reciprocity theorem for the chromatic polynomial. You are supposed to do only a few of the steps of this proof as a homework exercise, as indicated below. You are welcome do try the rest as an optional exercise, of course.

Throughout, $G = (V, E)$ denotes an undirected graph. A *k-coloring* of G is a map $c : V \to \{1, \ldots, k\}$. A k-coloring is *proper* if $c(v) \neq c(w)$ for all $v \neq w \in V$. The *chromatic polynomial* $\chi_G(k)$ is defined by

$$\chi_G(k) := \#\text{proper } k\text{-colorings of } G.$$

To show that this is in fact a polynomial in k, we realize χ_G as an Ehrhart polynomial as follows. We work in $\mathbb{R}^V$, i.e., we have one variable x_i for each vertex $i \in V$ of our graph. We define hyperplanes $H_{ij} = \{x \in \mathbb{R}^V \mid x_i = x_j\}$. The set of all these hyperplanes $B = \{H_{ij} | i, j \in V\}$ is called the *braid arrangement*, while the set of all these hyperplanes H_{ij} where ij is an edge of G is called the *graphic arrangement* A_G of G:

$$A_G = \{H_{ij} | ij \in E\}.$$

The first step in the proof is to show that $\chi_G(k-1)$ equals the Ehrhart function of the open 0,1-cube from which all points on hyperplanes H_ij with $ij \in E$ have been removed, i.e.,

$$\chi_G(k-1) = L_{(0,1)^V \setminus \bigcup_{H_{ij} \in A_G} H_{ij}}(k).$$

Note that the chromatic polynomial has been shifted by 1. (Here we use the following bijection between colorings and lattice points: A lattice point $x \in \mathbb{R}^V$ corresponds to the coloring c given by $c(v) = x_v$ for all $v \in V$.)

For everything that follows it is crucial to note that the set $(0,1)^V \setminus \bigcup_{H_{ij}\in A_G} H_{ij}$ is a disjoint union of open $|V|$-dimensional polytopes, i.e.,

$$(0,1)^V \setminus \bigcup_{H_{ij}\in A_G} H_{ij} = \bigcup_{i=1}^{n} \mathrm{relint}(P_i)$$

where the union is disjoint and the P_i are open $|V|$-dimensional polytopes which we call the *components*.

1) Visualize this construction as follows:

- Draw $k\cdot(0,1)^V \setminus \bigcup_{H_{ij}\in A_G} H_{ij}$ for $k=3$ and $G=(V,E)$ with $V=\{1,2,3\}$ and $E=\{12,23\}$.
- How many components are there?
- Locate the proper coloring given by $c(1)=2$, $c(2)=1$ and $c(3)=2$ in your drawing.
- Add $H_{13}\in B\setminus A_G$, the one hyperplane that is in the braid arrangement but not in the graphic arrangement, to your drawing.

We do not yet know that χ_G is a polynomial! To conclude this from Ehrhart's theorem we would need to know that the P_i are *lattice* polytopes, i.e, have vertices with integer coordinates. To get there we proceed as follows. The braid arrangement triangulates the cube $[0,1]^V$ into unimodular simplices. Because the graphical arrangement is a subarrangement of the braid arrangement, it follows that all the components $\mathrm{relint}(P_i)$ can be written as disjoint unions of open lattice simplices, whence the P_i are lattice polytopes.

2) Study the braid arrangement B:

- For $|V|=3$, $(0,1)^V\setminus\bigcup_{H_{ij}\in B} H_{ij}$ is the disjoint union of six 3-dimensional lattice polytopes, which are all simplices. What are the vertex sets of these six simplices?
- For one of these simplices, show that it is a lattice transform of the standard simplex Δ^3.
- In general, for $|V|=n$, what are the vertex sets of the components of $(0,1)^V\setminus\bigcup_{H_{ij}\in B} H_{ij}$? Can you write them down explicitly?

We now know that χ_G is a polynomial and we have a geometric representation of this polynomial. The central feature of this geometric representation $(0,1)^V\setminus\bigcup_{H_{ij}\in A_G} H_{ij}$ is that it decomposes into different connected components P_i. What is the *combinatorial* meaning of this?

With every proper coloring $x \in (0,1)^V \setminus \bigcup_{H_{ij} \in A_G} H_{ij}$ we associate an orientation $o(x)$ of G as follows. Let $vw \in E$ be any edge of E. vw is oriented from v towards w if $x_v < x_w$ and from w towards v if $x_w < x_v$. When analyzing this correspondence between proper colorings x and orientations $o(x)$, one comes to the following results.

- If x is a proper coloring, then $o(x)$ is acyclic. An orientation o of G is *acylic*, if *no* edge of G lies on a directed cycle w.r.t o.
- For every acyclic orientation o of G, there exists a coloring x such that $o(x) = o$.
- Two proper colorings x, y have the same acyclic orientation $o = o(x) = o(y)$ if and only if x and y lie in the same component P_i. We call this orientation o the orientation $o(P_i)$ of P_i.
- A (not necessarily proper) coloring x lies in P_i if and only if $x_v \leq x_w$ for all edges $vw \in E$ that are directed from v towards w in $o(P_i)$. In this case we call x and the orientation $o = o(P_i)$ *compatible*.

You are supposed to show the first of these statements.

> **3)** Prove that if x is a proper coloring, then $o(x)$ is acyclic.

Given this interpretation, we can now use Ehrhart-Macdonald reciprocity (a geometric result) to give a combinatorial interpretation of χ_G at negative integers. First, convince yourself that Ehrhart-Macdonald reciprocity implies:

$$|\chi_G(-(k+1))| = |L_{(0,1)^V \setminus \bigcup_{H_{ij} \in A_G} H_{ij}}(-k)|.$$

Then use this, and all other results above, for the following proof:

> **4)** Prove that for all $k > 1$, $|\chi_G(-(k+1))|$ equals the number of pairs (x, o) where x is a (not necessarily proper) $(k+1)$-coloring and a compatible acyclic orientation o.

Questions?

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