Using Intuitive Geometry - Exercise 6

Due Date: October 6th - Instructor: Felix Breuer

Exercises

1) Let $v_1, \ldots, v_n \in \mathbb{R}$ be real numbers. $m \in \mathbb{R}$ is a *median* of $v_1, \ldots, v_n$ if

$$\#\{i | v_i \leq m\} \geq \frac{n}{2} \leq \#\{i | v_i \geq m\}.$$

Your task is to show that the set of medians is the set of solutions to the following optimization problem:

$$\min_{m \in \mathbb{R}} \sum_{i=1}^{n} |v_i - m|.$$

Intuitively, this means that the median and the median hyperplanes are the "best fit" to the set of points $v_1, \ldots, v_n$, when minimizing the sum of distances. (As opposed to the sum of *squared* distances.)

2) The above optimization problem is not a priori a linear programming problem as the target function contains absolute values - and the absolute value is not a linear function. In this task, you are supposed to show that optimization problems that contain absolute values in the target function can be rewritten to obtain equivalent *linear* optimzation problems. To this end, proceed as follows.

Consider the optimization problem

$$m = \min\{\sum_{i=1}^{k} f_i(x) + \sum_{i=1}^{l} |g_i(x)| \mid Ax \leq b\}$$

where the f_i and g_i are *linear* functions. Assume that this problem has a feasible solution and that it is finite, i.e., assume that m is finite.

1. *Formulate* an optimization problem of the form

   $$\hat{m} = \min\{\sum_{i=1}^{\hat{k}} \hat{f}_i(x) + \sum_{i=1}^{l-1} |\hat{g}_i(x)| \mid \hat{A}x \leq \hat{b}\}$$

   with linear $\hat{f}_i$ and $\hat{g}_i$ such that the two problems have the same optimal value $\hat{m} = m$ and the formulation of $\hat{m}$ "uses one absolute value less", i.e., $l - 1 < l$. Note that the number $\hat{k}$ of linear functionals $\hat{f}_i$ may increase by more than one!
2. *Prove* your claim! That is, show that the problem you came up with does indeed have the same value as m.
3. Argue by induction that there exists a *linear* program with the same optimal value as m.

3) Let K be any simplicial complex such that all maximal faces of K have dimension d. Let

$\mathrm{sd}(K)$ be its barycentric subdivision. Show that there exists a labeling of the vertices of $\mathrm{sd}(K)$ with $1, \ldots, d+1$ such that every d-dimensional simplex $\sigma \in \mathrm{sd}(K)$ has the property that *all* labels appear on the vertices of σ.

More formally: Construct a function $p : \mathrm{sd}(K) \to \{1, \ldots, d+1\}$ such that for every d-dimensional simplex $\sigma \in K$, $p(V(\sigma)) = \{1, \ldots, d+1\}$ where $V(\sigma)$ denotes the vertex set of σ.

Hint: Try to solve the problem for the case $K = \Delta^d$ first and then argue that this implies the claim for general K. (Another hint: Draw a picture!)

4) The Theorem of PoincarÃ©-Miranda states the following:

> Let $f : [-1,1]^d \to \mathbb{R}^d$ be a continuous function such that for every face σ of the cube $[-1,1]^d$ and for every $a \in \{\pm e_1, \ldots, \pm e_d\}$,
>
> $$\sigma \subset H^+_{a,0} \Rightarrow f(\sigma) \subset H^+_{a,0}.$$
>
> Then there exists an $x \in [-1,1]^d$ that is a zero of f, i.e., $f(x) = 0$.

Show that the Theorem of PoincarÃ©-Miranda implies the following version of Brouwer's Fixed Point Theorem:

> Let $f : [-1,1]^d \to [-1,1]^d$ be a continuous function. Then there exists an $x \in [-1,1]^d$ that is a fixed point of f, i.e., $f(x) = x$.

Optional Problems

A) Show the converse of 4), i.e., prove that Brouwer's Fixed Point Theorem implies the Theorem of PoincarÃ©-Miranda.

B) Let $v_1, \ldots, v_n \in \mathbb{R}$ be real numbers and $\alpha \in [0,1]$. $q \in \mathbb{R}$ is an *α-quantile* of $v_1, \ldots, v_n$ if

$$\#\{i | v_i \leq q\} \geq \alpha \cdot n \text{ and } (1-\alpha) \cdot n \leq \#\{i | v_i \geq m\}.$$

Formulate an optimization problem whose set of optimal solutions is the set of α-quantiles of $v_1, \ldots, v_n$.

Questions?

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