Using Intuitive Geometry - Exercise 5

Due Date: September 29th - Instructor: Felix Breuer

Preliminaries

Recall that for $0 \neq a \in \mathbb{R}^d$

$$H_{a,b} = \{x \in \mathbb{R}^d \mid \langle a, x\rangle = b\}$$

$$H^+_{a,b} = \{x \in \mathbb{R}^d \mid \langle a, x\rangle \geq b\}$$

$$H^-_{a,b} = \{x \in \mathbb{R}^d \mid \langle a, x\rangle \leq b\}$$

Exercises

1) Let $H_{a,b}$ be any hyperplane with $a \neq 0$. Let $z \in H^-_{a,b}$ be any point with $\langle z, a\rangle < b$.

Show that there exsists a scalar $\lambda > 0$ such that for all $x \in H^+_{a,b}$ the Euclidean distance between $z + \lambda a$ and x is strictly smaller than the Euclidean distance between z and x, i.e,

$$||(z + \lambda a) - x|| < ||z - x||.$$

How large can you choose λ to be such that $z + \lambda a$ still has this property?

2) Let V be a set of n points in $\mathbb{R}^2$. For any $0 \neq a \in \mathbb{R}^2$, we define a *median hyperplane* of V in direction a to be a hyperplane of the form $H_{a,b}$ such that

$$|H^+_{a,b} \cap V| \geq \frac{n}{2} \leq |H^-_{a,b} \cap V|.$$

Prove that if n is odd, then for any a the median hyperplane of V in direction a is unique and that this hyperplane must contain a point in V.

3) The *core* of V is the set of all $x \in \mathbb{R}^2$ such that for all $z \in \mathbb{R}^2$, the number of $v \in V$ such that the distance between x and v is smaller or equal to the distance between z and v is greater than or equal to $\frac{n}{2}$. Assume that n is odd. Prove:

1. The core of V is contained in the intersection of all median hyperplanes.
2. The intersection of all median hyperplanes is contained in the core.

4) The point of this problem is to visualize the proof of the minimax theorem from Brouwer's fixed point theorem using the example of the "coin game" that we discussed in class. Throughout this problem, we are working with the *mixed extension* (!) of the two-person zero-sum game

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

. Your tasks are as follows:

1. Recall: What is the Nash equilibrium $(x^*, y^*) \in \Delta^{S_{Max}} \times \Delta^{S_{Mini}}$ of this game? (We had this in class.)
2. Given a strategy $x \in \Delta^{S_{Max}}$ of Max, what is the optimal response of Mini, i.e., what is the $y \in \Delta^{S_{Mini}}$ that minimizes

   $$\sum_{i,j} x_i y_j a_{ij}?$$

   Similarly, what is Max's optimal response x to a given strategy y of Mini?
3. Your solution of 2. determines two functions $f : \Delta^{S_{Max}} \to \Delta^{S_{Mini}}$ and $g : \Delta^{S_{Mini}} \to \Delta^{S_{Max}}$, where f maps a strategy of Max to Mini's best response and g maps a strategy of Mini to Max's best response. Plot these two functions, i.e., draw a graph! Note that in this case, we can identify $\Delta_{S_{Max}}$ and $\Delta_{S_{Mini}}$ with the interval $[-1, 1]$ (or, if you prefer, the interval [0,1]).
4. Consider the function $h : \Delta_{S_{Max}} \times \Delta_{S_{Mini}} \to \Delta_{S_{Max}} \times \Delta_{S_{Mini}}$ defined by $h(x, y) = (g(y), f(x))$. This is precisely the function h we used in the proof of the minimax theorem that maps any pair of strategies (x, y) to the pair of strategies that obtains if both players choose the best response to the strategy the opponent was using in the previous round! h can be viewed as a continuous map from a square into itself. Visualize this function by drawing it as a vector field on the square!
5. Observe that the Nash equilibrium is the unique sink of this vector field.

Optional Problems

A) Can you think of a way to prove some form of LP duality using Brouwer's fixed point theorem?

Questions?

eMail: felix@fbreuer.de - *web:* http://www.felixbreuer.net