Using Intuitive Geometry - Exercise 4

Due Date: September 22nd - Instructor: Felix Breuer

Announcement

I got the impression that you want to talk more about the exercises you have been working on than is possible in the last few minutes on Thursdays. Any suggestion how we could organize that differently are welcome!

Until then I want to stress: *I can make time to talk about anything you want to discuss!* If my current office hours don't fit your schedule, send me an email and we can work something out.

Preliminaries

Simplex Algorithm. Phase II of the Simplex Algorithm works as follows.

Given: a linear program $\max\{c|Ax \leq b\}$, a vertex x_0 and a set of d indices B, such that $Ax_0 \leq b$ and $A_B x_0 = b_B$.

Preparation: compute u with $uA = c$ and $u_i \neq 0$ only if $i \in B$.

Case 1: $u \geq 0$. Stop because x_0 is optimal.

Case 2: There exists $i \in B$ such that $u_i < 0$.

Compute y with $A_B y = -e_i$.

Case 2a: $Ay \leq 0$. Stop because the optimal value is unbounded.

Case 2b: There exists j with $a_j y > 0$.

Compute $\lambda = \min_j\{\frac{b_j - a_j x_0}{a_j y} | a_j y > 0\}$. Let j_0 be a j for which the minimal λ is attained. Then restart Phase II with $x_0 := x_0 + \lambda y$ and $B := B \cup \{j\} \setminus \{i\}$.

Game Theory. Please refer to the slides for game theoretic definitions. If something is still unclear afterwards, send me an email!

Note that we defined the "decision tree" we used in our description of Chess only very informally. So it is okay to use informal arguments about this tree in your homework.

LP Duality Theorem. Here is a very useful form of the LP Duality Theorem that you can use in your homework: If the following two LPs have a feasible solution then their optimal values are equal.

Primal LP:

$$
\begin{aligned}
\max cx + dy& \\
Ax + By &\leq a \\
Cx + Dy &= b \\
x &\geq 0
\end{aligned}
$$

Dual LP:

$$
\begin{aligned}
\min ua + vb& \\
uA + vC &\geq c \\
uB + vD &= d \\
u &\geq 0
\end{aligned}
$$

Exercises

1) You have already met the polytope P defined by

$$
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \cdot x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \end{pmatrix}.
$$

The goal is to optimize the target function given by $c = (1,0,0)$ over this polytope. To this end you are supposed to compute **one step** in Phase II of the simplex algorithm, by applying Phase II to

$$x_0 = (0,1,0) \text{ and } B = \{1,2,5\}.$$

2) Let $A = (a_{ij})$ be a real matrix with rows $i = 1,\ldots,n$ and columns $j = 1,\ldots,m$. Let $S = \{1,\ldots,m\}$ and let Δ^S denote the $(m-1)$-dimensional probability simplex

$$\Delta^S = \{y \in \mathbb{R}^m \mid y_j \geq 0, \sum_j y_j = 1\}.$$

Moreover, let $x \in \mathbb{R}^n, x_i \geq 0, \sum_i x_i = 1$. Show that

$$\min_{j \in S} \sum_i x_i a_{ij} = \min_{y \in \Delta^S} \sum_{i,j} x_i y_j a_{ij}.$$

3) Show that for any zero-sum two person game

$$\max_s \min_y K(x,y) \leq \min_y \max_x K(x,y),$$

that is, the payoff for Max in his worst case analysis is less than or equal than the payoff for Mini in her worst case analysis.

4) Consider the game of Chess, with the additional assumption that all Chess games are finite. (Say, there is a rule that says games that last longer than 10^{100} moves result in an automatic tie.)

- Show that Chess has a Nash equilibrium in pure strategies. (Hint: Use induction over subtrees of the decision tree.)
- Show that *either* there exists a pure strategy for White such that White wins, no matter how Black plays, *or* there exists a pure strategy for Black such that Black can win or tie, no matter how White plays. (Hint: Apply the fact that $\min_y \max_x K(x,y) = \max_x \min_y K(x,y)$ for games that have a Nash equilibrium.)

Note:

- This is saying that Chess is trivial, because one of the two players, in essence, has a winning strategy!
- Even though the above proof of this claim is constructive, nobody can actually use this strategy, because in practice, it is impossible to even write down the entire decision tree for Chess!
- The above proof works for all games in which both players have "complete information".

5) Complete the proof of the Minimax Theorem: Using the form of the LP Duality Theorem given in the preliminaries, show that the following two LPs have equal optimal values.

$$\begin{aligned} & \max\{c \in \mathbb{R} \mid c \leq \sum_i x_i a_{ij}, \sum_i x_i = 1, x_i \geq 0\} \\ = \; & \min\{c \in \mathbb{R} \mid c \geq \sum_j y_j a_{ij}, \sum_j y_j = 1, y_j \geq 0\} \end{aligned}$$

Optional Problems

A) Prove that the version of LP Duality given in the preliminaries is equivalent to the one we had in class.

B) Prove that the Minimax Theorem implies LP Duality.

C) Prove that for zero-sum two person games, the existence of Nash equilibria is equivalent to

$$\max_s \min_y K(x,y) = \min_y \max_x K(x,y).$$

Questions?

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