Using Intuitive Geometry - Exercise 2

Due Date: September 8th - Instructor: Felix Breuer

Preliminaries

Posets. A *poset* is a pair $(S, \leq)$ of a set S and a binary relation $\leq$ with the following properties:

- reflexivity: $x \leq x$ for all $x \in S$.
- transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in S$.
- antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in S$.

Let $(S, \leq)$ be a poset. A *minimal element* is an element $x \in S$ such that $x \leq y$ for all $y \in S$. A *maximal element* is an element $x \in S$ such that $y \leq x$ for all $y \in S$. An *upper bound* of $x, y \in S$ is an element $z \in S$ such that $x, y \leq z$. An *lower bound* of $x, y \in S$ is an element $z \in S$ such that $z \leq x, y$. A *unique least upper bound* of $x, y \in S$ is an upper bound $z \in S$ of x and y with the additional property that for every upper bound $z' \in S$ of x and y we have $z \leq z'$. A *unique greatest lower bound* of $x, y \in S$ is a lower bound $z \in S$ of x and y with the additional property that for every lower bound $z' \in S$ of x and y we have $z' \leq z$.

Polytopes. A V-polytope is a set $P \subset \mathbb{R}^d$ of the form

$$P = \mathrm{conv}(v_1, \ldots, v_k) = \{\sum_{i=1}^{n} \lambda_i v_i | \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0\}.$$

An H-polytope is a *bounded* (!) set $P \subset \mathbb{R}^d$ of the form

$$P = \{x \in \mathbb{R}^d | Ax \leq b\},$$

for some matrix A and vector b. The fundamental theorem of polytope theory states that

> every H-polytope is a V-polytope and every V-polytope is an H-polytope.

The *Minkowski sum* of two sets $A, B \subset \mathbb{R}^d$ is

$$A + B = \{a + b | a \in A, b \in B\}.$$

Exercises

1) Let $(S, \leq)$ be a poset that has a minimal element 0 and a maximal element 1 and the property that every $x, y \in S$ have a unique greatest lower bound in S. Show that every $x, y \in S$ have a unique least upper bound.

2) Using the fundamental theorem of polytope theory, show that the following statements:

- The intersection of a polytope with an affine subspace is a polytope.
- The intersection of a polytope with a polytope is a polytope.

- The Minkowski sum of two polytopes is a polytope.
- If $f: \mathbb{R}^d \to \mathbb{R}^m$ is a linear map and P is a polytope, then $f(P)$ is a polytope.

Note that some of these are significantly easier to prove with the H-description and some are easiert to prove with the V-description of a polytope.

3) What are the vertices of the polytope given by the following inequality description?

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \cdot x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \end{pmatrix}$$

Optional Problems

A) We have represented the vertices of the permutahedron on the one hand as permutations of the numbers $1, \ldots, n$, on the other hand as linear orderings of the variables $x_1, \ldots, x_n$, so that, for example the vertex $(1,3,4,2,5)$ corresponds to $x_1|x_4|x_2|x_3|x_5$. (Note that 13425 is the inverse of the permutation 14235.) Show that, in the second representation, two vertices are adjacent (lie on a common edge) if and only if they differ by an adjacent transposition (swapping two variables that are next to each other).

Questions?

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