Using Intuitive Geometry - Exercise 1

Due Date: September 1st - Instructor: Felix Breuer

Preliminaries

A *lattice transformation* $f : \mathbb{R}^d \to \mathbb{R}^d$ is an affine isomorphism that, when restricted to the integer lattice $\mathbb{Z}^d$, induces a bijection (i.e., a 1-to-1 and onto map) on the integer lattice $f|_{\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{Z}^d$.

A *lattice basis* is a set $a_1, \ldots, a_d \in \mathbb{Z}^d$ of linearly independent integer vectors such that for every $z \in \mathbb{Z}^d$ there exist *integers* $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$ such that

$$\sum_{i=1}^{d} \lambda_i a_i = z.$$

The *fundamental parallelepiped* $\Pi_{a_1,\ldots,a_d}$ spanned by these vectors is the set

$$\Pi_{a_1,\ldots,a_d} = \{x \in \mathbb{R}^d \mid x = \sum_{k=1}^{d} \lambda_i a_i \text{ for } 0 \leq \lambda_i < 1\}.$$

In these exercises you may use the following:

Theorem. Let $a_1, \ldots, a_d \in \mathbb{Z}^d$ be linearly independent integer vectors. Let $A = (a_1 \ \ldots \ a_d)$ be the matrix with the a_i as columns. Then

$$L(\Pi_{a_1,\ldots,a_d}) = \text{Volume}(\Pi_{a_1,\ldots,a_d}) = |\det(A)|.$$

Exercises

1) Show that for an affine isomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form $f(x) = Ax + b$ the following are equivalent:

- f is a lattice transformation.
- $L(X) = L(f(X))$ for *every* set $X \subset \mathbb{R}^d$.
- A is an integer matrix, b an integer vector and $|\det(A)| = 1$.

2) Show that for linearly independent integer vectors $a_1, \ldots, a_d \in \mathbb{Z}^d$ the following are equivalent:

- $a_1, \ldots, a_d$ form a lattice basis.
- There exists a lattice transformation f such that $f(a_i) = e_i$ for $i = 1, \ldots, d$ where e_i are the standard unit vectors.
- The fundamental parallelepiped $\Pi_{a_1,\ldots,a_d}$ contains exactly one lattice point $z \in \mathbb{Z}^d$.

Optional Problems

A) In dimension *2*, show the theorem about the fundamental parallelepiped stated in the preliminaries (i.e., $L(\Pi_{a,b}) = \text{Volume}(\Pi_{a,b})$) *without* using limit arguments.

B) Let $\gcd(a,b)=1$. Is there a simple geometric argument that shows

$$\sum_{k=0}^{b-1}(ka \bmod b)\cdot k + \sum_{k=0}^{a-1}(kb \bmod a)\cdot k = \text{something simple?}$$

This is asking for an intuitive geometric proof of Dedekind reciprocity.

C) If a, b, c are pairwise relatively prime, we have shown

$$\sum_{k=0}^{c-1}\lfloor\frac{ka}{c}\rfloor\cdot\lfloor\frac{kb}{c}\rfloor + \sum_{k=0}^{b-1}\lfloor\frac{ka}{b}\rfloor\cdot\lfloor\frac{kc}{b}\rfloor + \sum_{k=0}^{a-1}\lfloor\frac{kc}{a}\rfloor\cdot\lfloor\frac{kb}{a}\rfloor = (a-1)(b-1)(c-1).$$

Does this imply a reciprocity law for Dedekind sums or sums of the form

$$\sum_{k=0}^{c}\{\frac{ka}{c}\}\cdot\{\frac{kb}{c}\}?$$

Questions?

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