Def Let $v_1, \dots, v_k \in \mathbb{R}^d$.

The *linear* hull of the v_i is

$$\mathrm{lin}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

The *affine* hull of the v_i is

$$\mathrm{aff}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R},\ \sum \lambda_i = 1 \right\}$$

The *conical* hull of the v_i is

$$\mathrm{cone}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R},\ \lambda_i > 0 \right\}$$

The *convex* hull of the v_i is

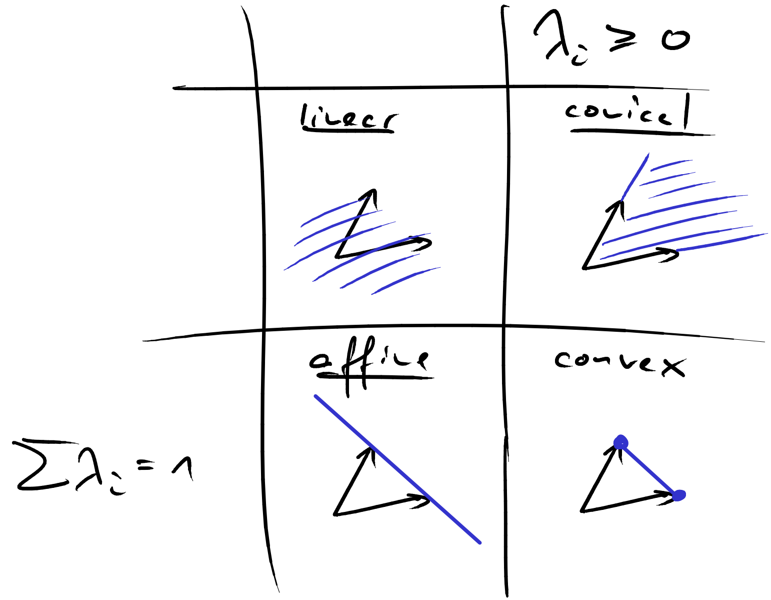
$$\mathrm{conv}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R},\ \sum \lambda_i = 1 \ \underline{\text{and}}\ \lambda_i \geq 0 \right\}$$

Given $v_1, \dots, v_k \in \mathbb{R}^d$, the

$$\left\{ \begin{matrix} \text{linear}, & \text{affine}, \\ \text{conical}, & \text{convex} \end{matrix} \right\} \text{ hull}$$

of $v_1, \dots, v_k$ is the set

$$\left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R} \text{ and } \dots \right\}.$$



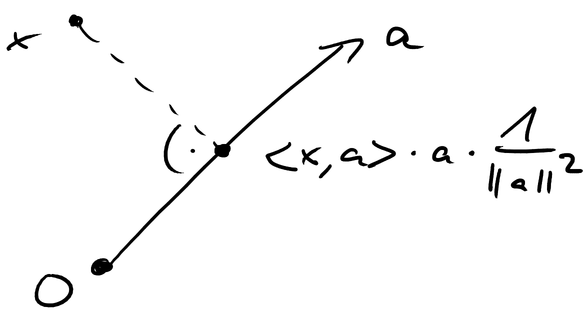
Hyperplanes

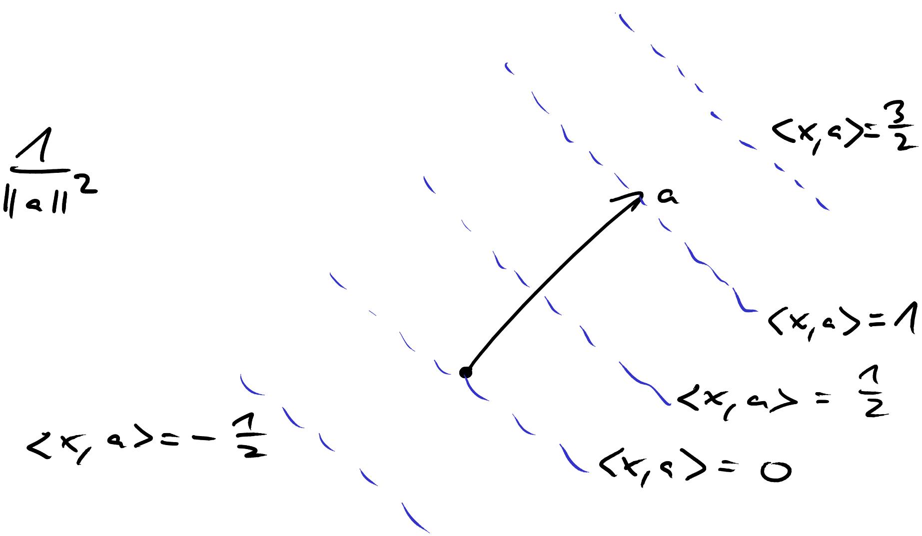
A hyperplane in $\mathbb{R}^d$ is an affine subspace of dimension $d-1$.

Fact: For every hyperplane $H \subset \mathbb{R}^d$ there exist $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$H = \{ x \in \mathbb{R}^d \mid \langle x, a \rangle = b \}$$

If $\|a\| = 1$:





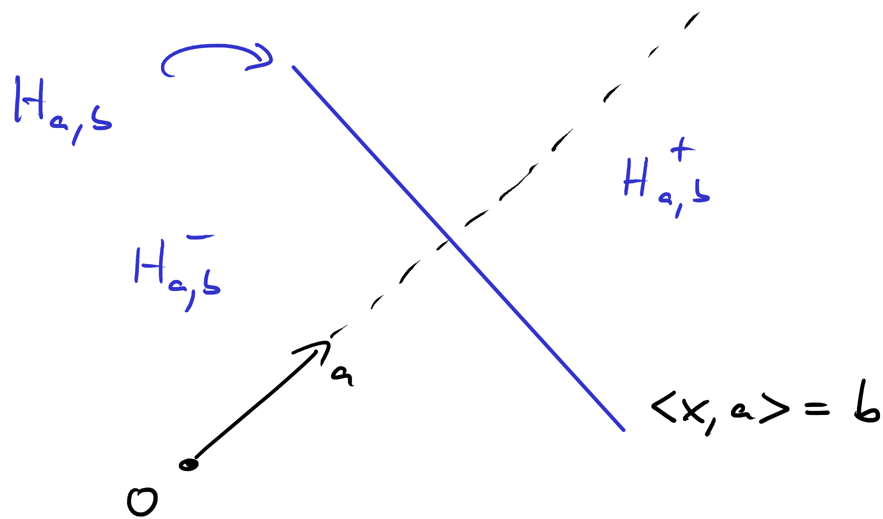
We write for $a \in \mathbb{R}^d$, $b \in \mathbb{R}$:

$$H_{a,b} = \{ x \in \mathbb{R}^d \mid \langle x, a \rangle = b \}$$ hyperplane

$$H^+_{a,b} = \{ x \in \mathbb{R}^d \mid \langle x, a \rangle \geq b \}$$

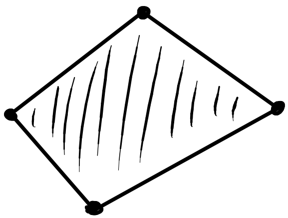
half-space

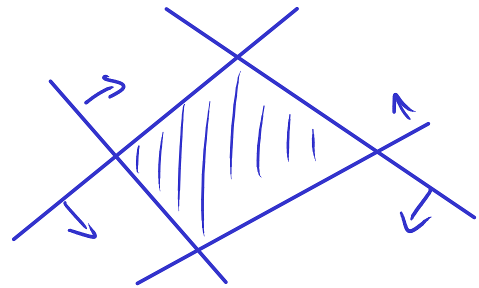
$$H^-_{a,b} = \{ x \in \mathbb{R}^d \mid \langle x, a \rangle \leq b \}$$



A **V-polytope** is the convex hull of finitely many points.

An **H-polytope** is an intersection of finitely many (closed) half-spaces, that is **bounded**.





$P = \mathrm{conv}(v_1, \ldots, v_n)$

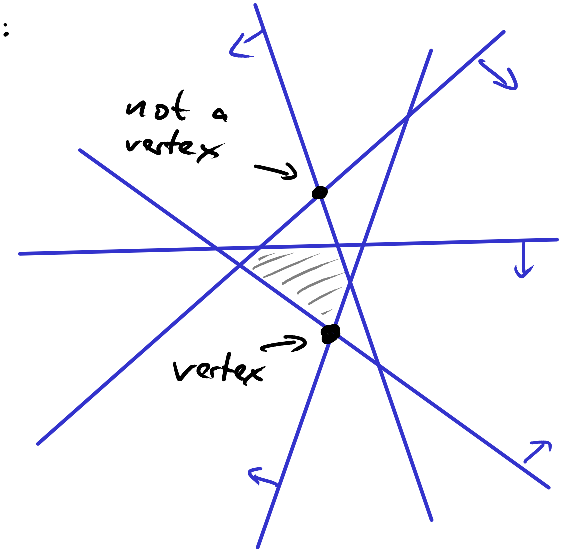
$P = \bigcap_i H^+_{a_i, b_i} = \{x \mid Ax \geq b\}$

Theorem Every H-polytope is a V-polytope and every V-polytope is an H-polytope.

Intuition

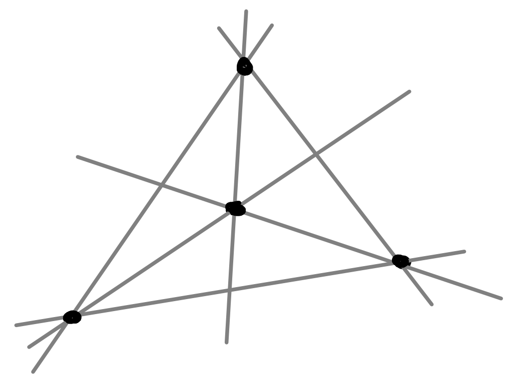
Convert $H \to V$:

In $\mathbb{R}^d$, the intersection of any d (indep.) hyperplanes will yield a point p. p is a vertex, if it is not cut-off by any other hyperplane.



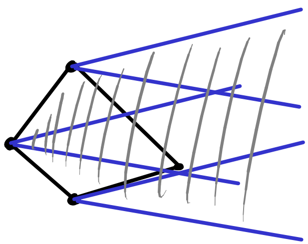
Convert $V \to H$

In $\mathbb{R}^d$, d (indep) points span an affine hyperplane H. If H^+ or H^- contain <u>all</u> points, then H^+ (resp H^-) determines a defining inequality.



A **V-polyhedron** is a set of the form

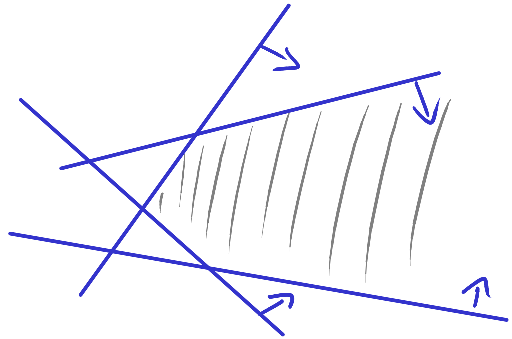
$$P = \mathrm{conv}(v_1, \dots, v_k) + \mathrm{cone}(a_1, \dots, a_\ell)$$



Minkowski sum $A+B = \{a+b \mid a\in A, b\in B\}$

An **H-polyhedron** is the intersection of finitely many (closed) half-spaces.

$$P = \bigcap_i H^+_{a_i, b_i} = \{x \mid Ax \geq b\}$$



Theorem Every H-polyhedron is a V-polyhedron and every V-polyhedron is an H-polyhedron.

$\mathrm{conv}(v_1,\dots,v_k) + \mathrm{cone}(a_1,\dots,a_\ell)$ is bounded

$$\Longleftrightarrow \mathrm{cone}(a_1,\dots,a_\ell) = \{0\} \quad (\text{i.e., } \ell = 0)$$

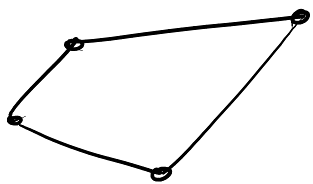
Thm 2 => Thm 1

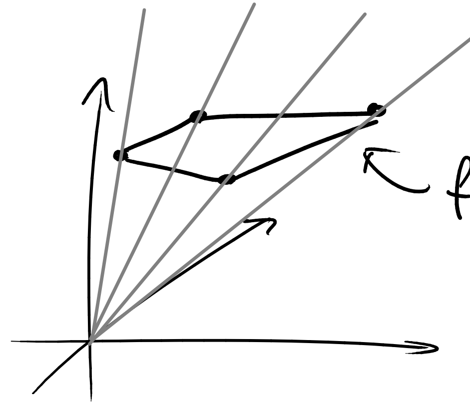
An H-polytope is a bounded H-polyhedron
is a bounded V-polyhedron
is a V-polytope.

A V-polytope is a bounded V-polyhedron
is a bounded H-polyhedron
is an H-polytope.

Homogenization

$\mathbb{R}^2 \supset P$





$\mathbb{R}^3 \supset P'$

$f(P)$

<u>Idea:</u> Given $P \subset \mathbb{R}^2$.

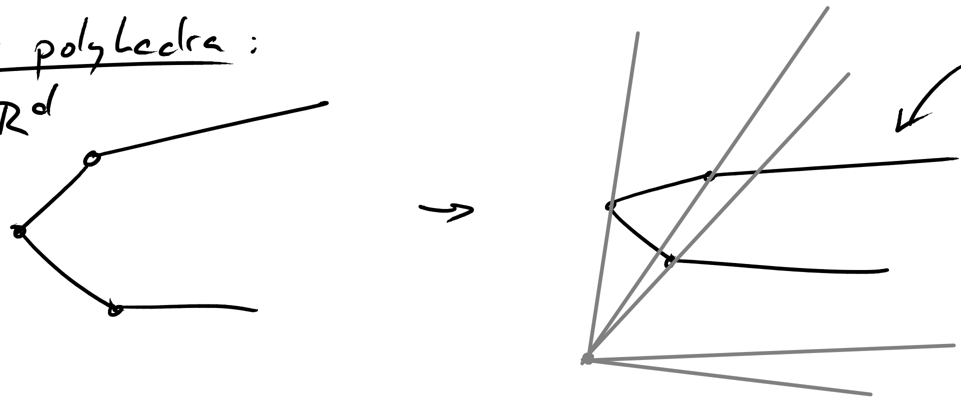
Embed $f: \mathbb{R}^2 \hookrightarrow \{x \in \mathbb{R}^3 : x_3 = 1\}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\underbrace{\operatorname{cone}(f(P))}_{=P'} \cap \{x \in \mathbb{R}^3 \mid x_3 = 1\} = f(P)$$

For polyhedra:

$P \subset \mathbb{R}^d$ $\longrightarrow$ $f(P)$ $\quad P' \subset \mathbb{R}^{d+1}$



Converting V-representations

$P = \operatorname{conv}(V) + \operatorname{cone}(Y) \longrightarrow P' = \operatorname{cone}\begin{pmatrix} V & Y \\ 1 & 0 \end{pmatrix}$

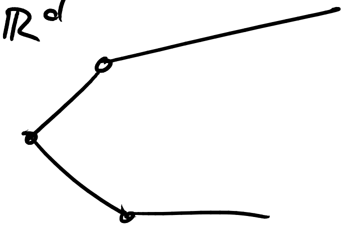
$$f(P) = \left\{ \binom{x}{1} \in \mathbb{R}^{d+1} \;\middle|\; \sum \alpha_i v_i + \sum \beta_i y_i = x,\ \alpha_i, \beta_i \geq 0,\ \sum \alpha_i = 1 \right\}$$

$$= \left\{ x' \in \mathbb{R}^{d+1} \;\middle|\; \sum \alpha_i \binom{v_i}{1} + \sum \beta_i \binom{y_i}{0} = x',\ \alpha_i, \beta_i \geq 0,\ x'_{d+1} = 1 \right\}$$

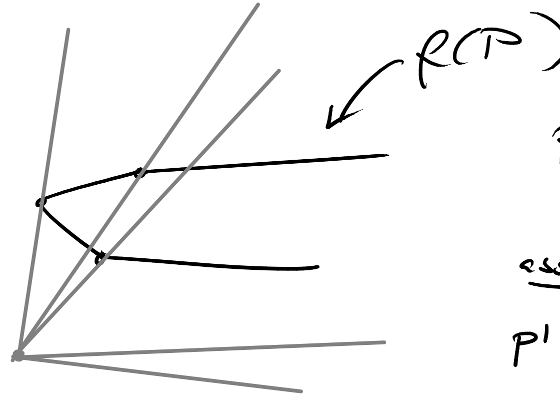
$$= P' \cap \{ x \mid x'_{d+1} = 1 \}$$

For polyhedra:

$P \subset \mathbb{R}^d$



$\to$



$P' \subset \mathbb{R}^{d+1}$

assume

$P' \subset \{x \mid x_{d+1} \geq 0\}$

Converting V-representations

$P = \mathrm{conv}(\hat{X}') + \mathrm{cone}(\hat{X}'') \longleftarrow P' = \mathrm{cone}(X)$

where $X = \underbrace{x_1', \dots, x_e'}_{x'_{i(d+1)} > 0}, \underbrace{x_1'', \dots, x_m'}_{x'_{i(d+1)} = 0}$

Define $\hat{x}_i'$ by $\begin{pmatrix} \hat{x}_i' \\ 1 \end{pmatrix} = \frac{x_i'}{x'_{i(d+1)}}$

$\hat{x}_i''$ by $\begin{pmatrix} \hat{x}_i'' \\ 0 \end{pmatrix} = x_i''$

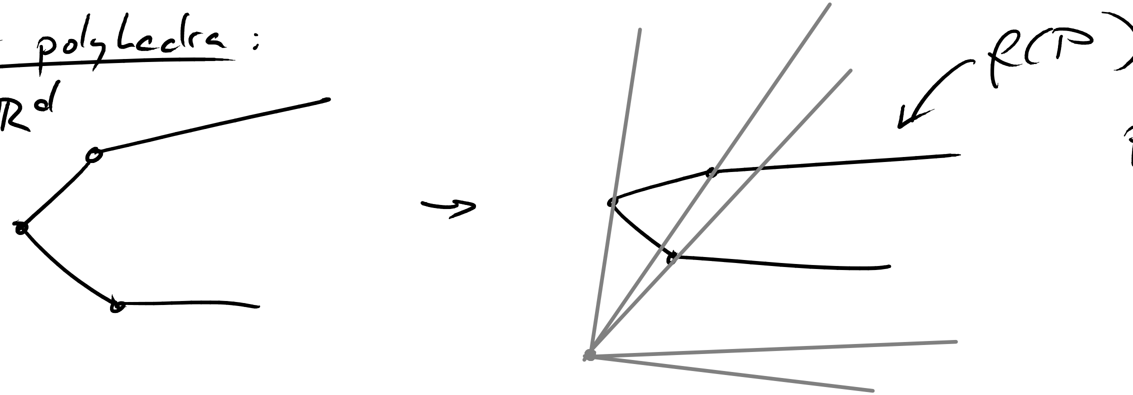
$$f(P) = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1} \,\middle|\, \sum \alpha_i \hat{x}_i' + \sum \beta_i \hat{x}_i'' = x,\ \alpha_i, \beta_i \geq 0,\ \sum \alpha_i = 1 \right\}$$

$$= \left\{ y \in \mathbb{R}^{d+1} \,\middle|\, \sum \alpha_i \begin{pmatrix} \hat{x}_i' \\ 1 \end{pmatrix} + \sum \beta_i \begin{pmatrix} \hat{x}_i'' \\ 0 \end{pmatrix} = y,\ \alpha_i, \beta_i \geq 0,\ y_{d+1} = 1 \right\}$$

$$= \left\{ y \in \mathbb{R}^{d+1} \,\middle|\, \sum \left(\alpha_i \frac{1}{x'_{i(d+1)}}\right) x_i' + \sum \beta_i x_i'' = y,\ \alpha_i, \beta_i \geq 0 \right\} \cap \{ y \mid y_{d+1} = 1 \}$$

For polyhedra:

$P \subset \mathbb{R}^d$



$P' \subset \mathbb{R}^{d+1}$

Converting H-representations

$$P = \{x \mid Ax \geq b\} \longrightarrow P' = \{x' \mid (A \; -b)x' \geq 0\}$$

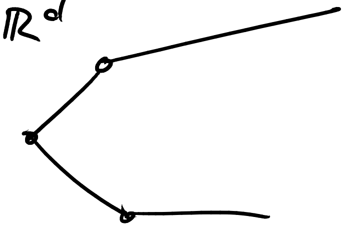
$$f(P) = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \,\middle|\, Ax \geq b \right\}$$

$$= \{x' \in \mathbb{R}^{d+1} \mid Ax' - b x'_{d+1} \geq 0, \; x'_{d+1} = 1\}$$

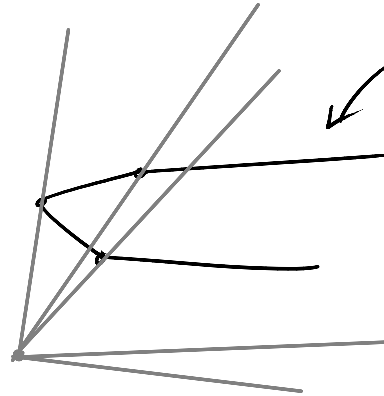
$$= P' \cap \{x' \mid x'_{d+1} = 1\}$$

For polyhedra:

$P \subset \mathbb{R}^d$



$\rightarrow$



$P' \subset \mathbb{R}^{d+1}$

$A = (a_1, \dots, a_{d+1})$

Converting H-representations

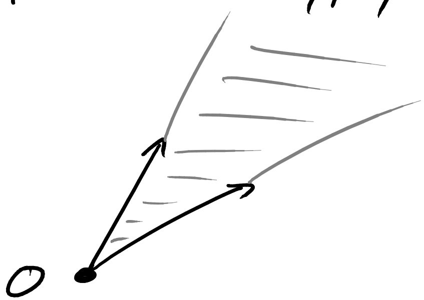
$$P = \{ x \mid (a_1, \dots, a_d) x \geq -a_{d+1} \} \longleftarrow P' = \{ x' \mid A x' \geq 0 \}$$

$$P' \cap \{ x' \mid x'_{d+1} = 1 \} = \{ \binom{x}{1} \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d} a_i x_i + a_{d+1} \geq 0 \}$$

$$= f(P)$$

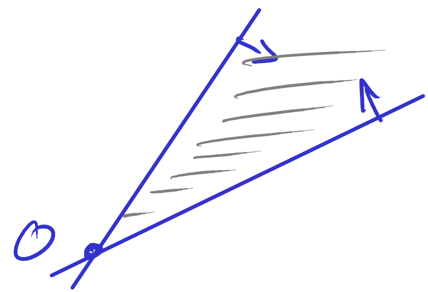
A **V-cone** is the conical hull of finitely many vectors.

$$P = \operatorname{cone}(v_1, \dots, v_k)$$



An H-cone is the intersection of finitely many (closed) **linear** hyperplanes.

$$P = \bigcap_i H^+_{a_i, 0}$$



Theorem Every H-cone is a V-cone and every V-cone is an H-cone.

Thm 3 => Thm 2

Given P_V in V-representation.

- there exists P_V' in V-rep with $f(P_V) = P_V' \cap \{x \mid x_{d+1} = 1\}$.
- by Thm 3, P_V' has an H-rep P_H'.
- there exists P_H in H-rep with $f(P_H) = P_H' \cap \{x \mid x_{d+1} = 1\}$

$$f(P_V) = P_V' \cap \{x \mid x_{d+1} = 1\} = P_H' \cap \{x \mid x_{d+1} = 1\} = f(P_H).$$

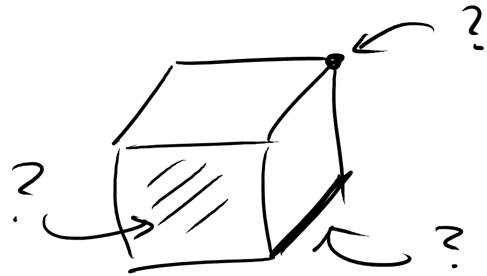
Given P_H in H-representation.

- there exists P_H' in H-rep with $f(P_H) = P_H' \cap \{x \mid x_{d+1} = 1\}$
- by Thm 3, P_H' has a V-rep P_V'.
- there exists P_V in V-rep with $f(P_V) = P_V' \cap \{x \mid x_{d+1} = 1\}$

! as P_H' has $x_{d+1} \geq 0$, all generators of P_V' have $v_{d+1} \geq 0$!

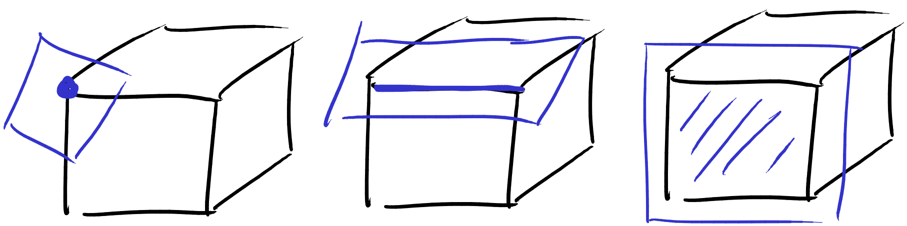
$$f(P_H) = P_H' \cap \{x \mid x_{d+1} = 1\} = P_V' \cap \{x \mid x_{d+1} = 1\} = f(P_V) \quad \square$$

Faces of Polytopes



An inequality $\langle x, a \rangle \geq b$ is **valid** if it holds for all $x \in P$, i.e., if $H^+_{a,b} \supset P$.

A **face** of P is the intersection $H_{a,b} \cap P$ of P with the hyperplane corresponding to a valid inequality $\langle x, a \rangle \geq b$.

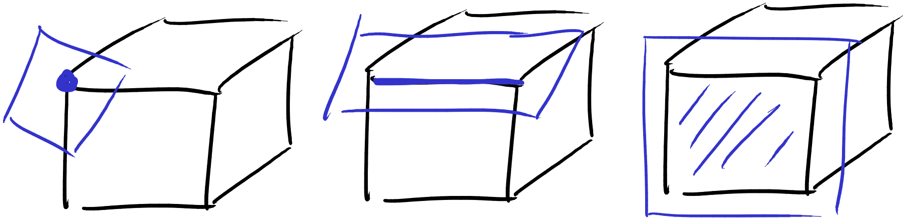


P is a face of P for $0 \cdot x \geq 0$.

$\emptyset$ is a face of P for $0 \cdot x \geq 1$

(degenerate hyperplanes)

The **dimension** of a face is the dimension of its affine hull: $\dim(\mathrm{aff}(H_{a,\delta} \cap P))$.



dim 0 dim 1 dim 2

$$\dim(P) = \dim(\mathrm{aff}(P)) = 3$$

$$\dim(\emptyset) = -1$$

If $\dim P = d$, faces of various dimension have special names:

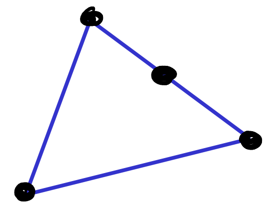
dimension of F :	0	1	$d-2$	$d-1$
name of F :	vertex	edge	ridge	facet

the corresponding hyperplanes are called "facet-defining"

Facts about Faces and Vertices

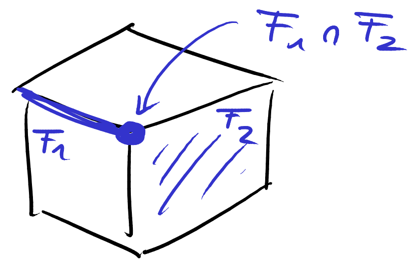
1) $P = \mathrm{conv}(\mathrm{vert}(P))$

but $V \supsetneqq \mathrm{vert}(\mathrm{conv}(V))$



2) If F_1, F_2 are faces of P,

then $F_1 \cap F_2$ is a face of P.



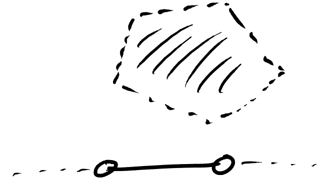
3) If F is a face of P,

then F is a polytope.

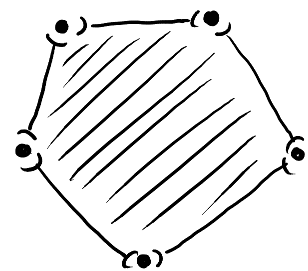
4) $\mathrm{vert}(F) = F \cap \mathrm{vert}(P)$

5) F' is a face of $F \iff F' \subset F$ and F' is a face of P

The *relative interior* $\mathrm{relint}(P)$ of P is the interior of P relative to $\mathrm{aff}(P)$



Lemma For every $y \in P$ there is a unique face F of P with $y \in \mathrm{relint}(F)$.



1) $F = P \cap \bigcap H_{a,b}$

$H_{a,b}$ is a facet-defining hyperplane with $\langle y, a \rangle = b$.

2) $\mathrm{vert}(F) = \{ v \in \mathrm{vert}(P) \mid$ there exists a representation

$$y = \sum_{v' \in \mathrm{vert}(P)} \lambda_{v'} v', \quad \lambda_{v'} \geq 0, \quad \sum_{v'} \lambda_{v'} = 1$$

with $\lambda_v > 0 \}$

Posets

A poset (partially ordered set) is a pair $(S, \leq)$ where S is a set and $\leq$ is a binary relation that is

reflexive: $x \leq x$

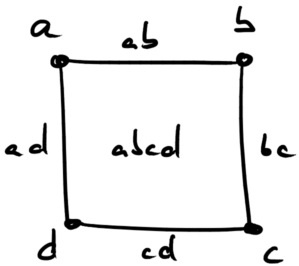
transitive: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

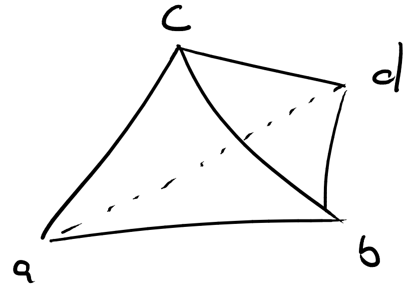
antisymmetric: $x \leq y \wedge y \leq x \Rightarrow x = y$

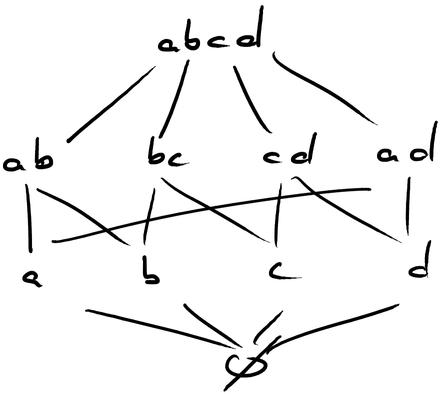
Lemma: $(\{F \mid F \text{ is a face of } P\}, \subset)$ is a poset called the <u>face lattice</u> of P.

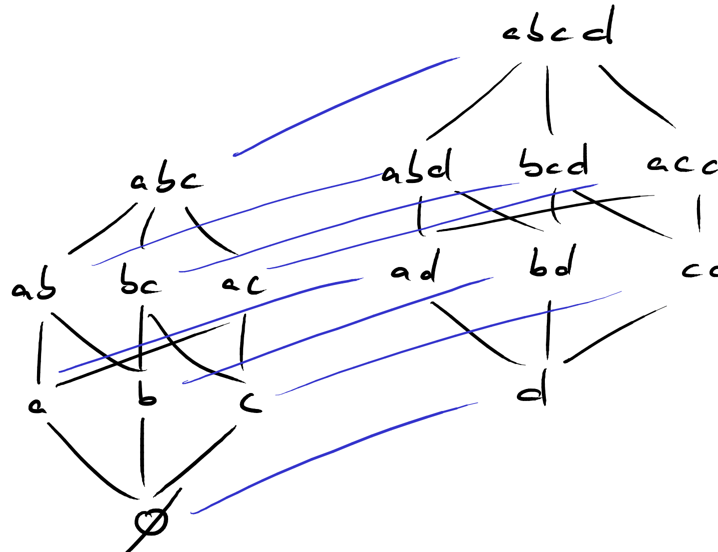
this "lattice" has nothing to do with "lattice points"

Proof: $(S, \subset)$ is <u>always</u> a poset if "$\subset$" is set-inclusion.

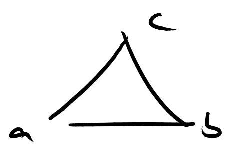


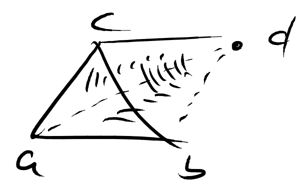


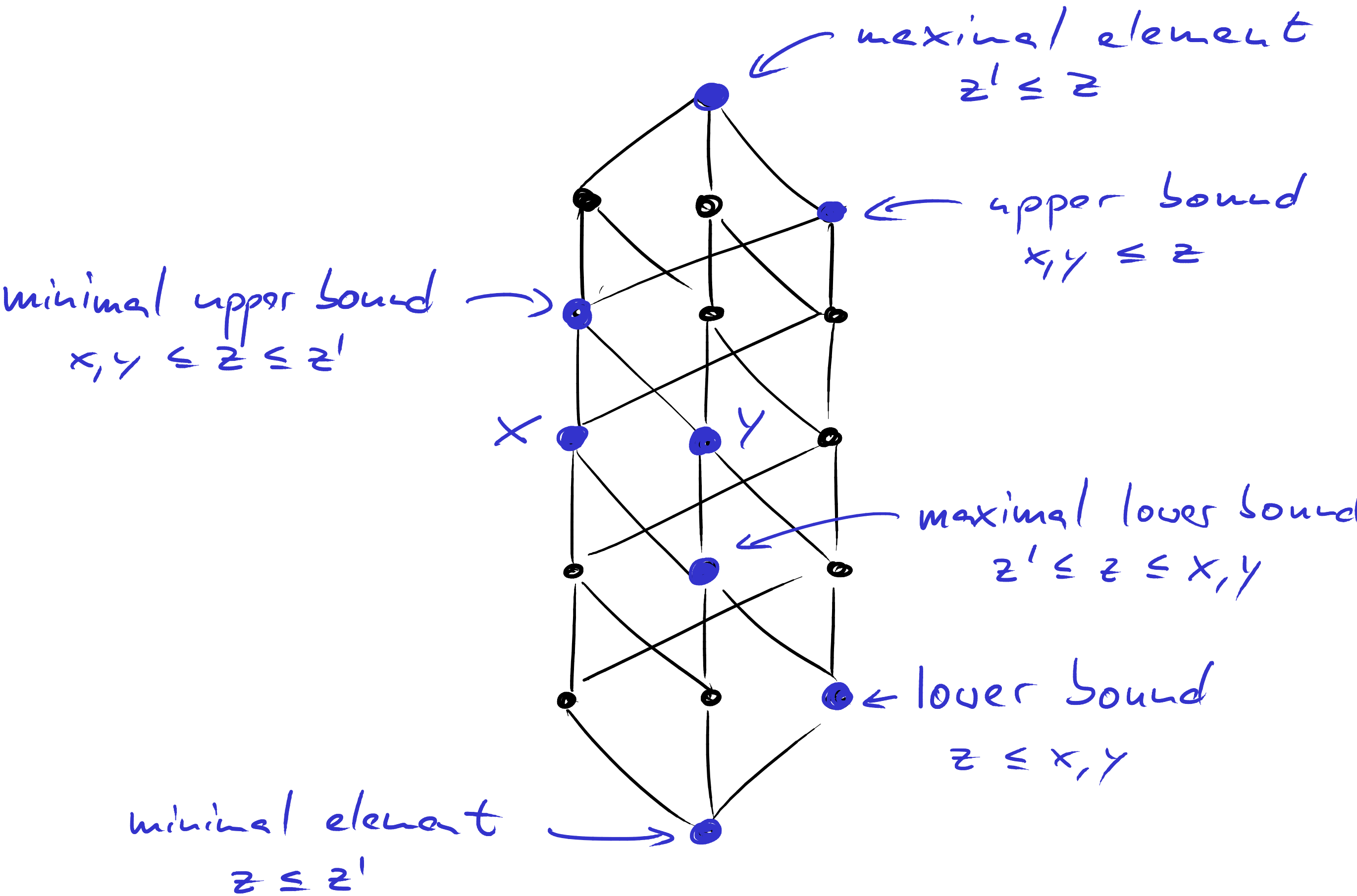




Hesse Diagram.






maximal element
z' ≤ z
upper bound
x, y ≤ z
minimal upper bound
x, y ≤ z ≤ z'
x
y
maximal lower bound
z' ≤ z ≤ x, y
lower bound
z ≤ x, y
minimal element
z ≤ z'

A <u>lattice</u> is a poset $(S, \leq)$ such that

1) there exist a unique minimal element and a unique maximal element.
2) any $x, y \in S$ have a unique maximal lower bound
3) any $x, y \in S$ have a unique minimal upper bound

<u>Theorem</u> $(\{F \mid F \text{ face of } P\}, \leq)$ is a lattice.

<u>Proof</u>: 1) $\emptyset$ is unique min elt, P is unique max elt.

2) $F_1 \cap F_2$ is unique maximal lower bound.

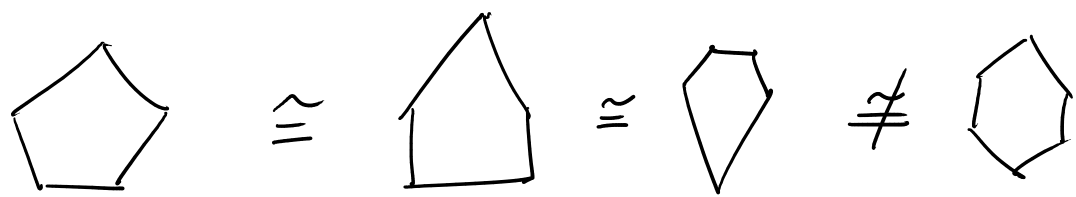
3) Is implied by 1) and 2) !

A *poset isomorphism* from $(S, \leq)$ to $(S', \leq')$ is a bijection $f: S \longrightarrow S'$ such that

$$x \leq y \iff f(x) \leq' f(y).$$

$(S, \leq)$ and $(S', \leq')$ are *isomorphic* if there exists a poset isomorphism from one to the other.

Def: Polytopes P and P' are *combinatorially equivalent* if their face lattices are isomorphic.



Permutahedron

A *permutation* of n elements is a bijection $\pi : [n] \to [n]$.
Equivalently, it is a linear ordering of $\{1, \dots, n\} =: [n]$.

Permutations of $\{1,2,3\}$: 123, 132, 213, 231, 312, 321

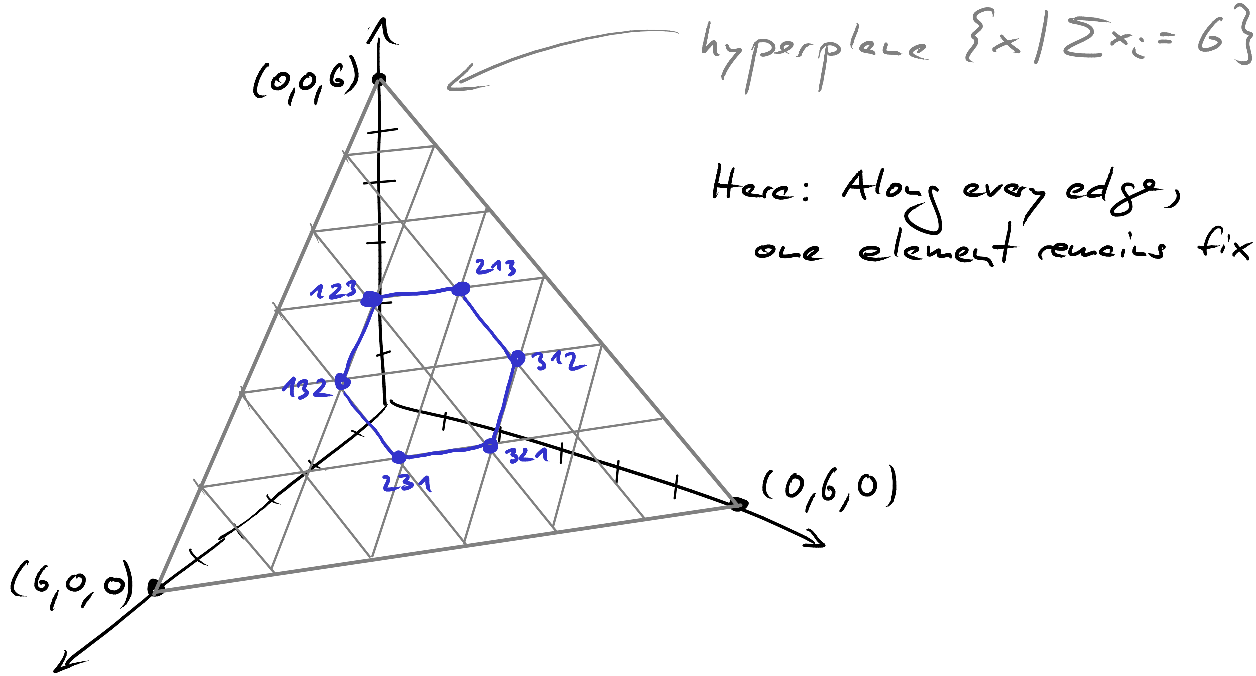
The *permutahedron* is

$$\Pi_n := \operatorname{conv} \left\{ \begin{pmatrix} \pi(1) \\ \vdots \\ \pi(n) \end{pmatrix} \;\middle|\; \pi : [n] \to [n] \text{ a permutation} \right\}.$$

$$\Pi_3 = \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$$

How does it look like?

Observation: For all $x \in \Pi_n$, $\sum x_i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.



Here: Along every edge, one element remains fix.

Π_4 lives in $\mathbb{R}^4$, but it is 3 dimensional, so we can draw it...

What is an H-representation of Π_n?

<u>Idea:</u> Partition $[n] = A \dot{\cup} B$.

$$\Pi_n \subset \left\{ x \;\middle|\; \sum_{i \in A} x_i \geq \text{minimal} = \frac{|A|(|A|+1)}{2} \right\}$$

$$\Pi_n \subset \left\{ x \;\middle|\; \sum_{i \in A} x_i \leq \text{maximal} = \frac{|A|(|A|+1)}{2} + |A|(n-|A|) \right\}$$

$$= \left\{ x \;\middle|\; \sum_{i \in B} x_i \geq \text{minimal} \right\}$$

Theorem: The facet-defining inequalities of Π_n are $\sum_{i\in A} x_i \geq \frac{k(k+1)}{2}$ where $A \in \binom{[n]}{k}$, $k \in [n-1]$.

$A \subset [n]$ with $|A| = k$

Proof: Let $T = \{ x \mid \sum_{i\in A} x_i \geq \frac{k(k+1)}{2},\ A \in \binom{[n]}{k}, k \in [n-1], \sum_{i\in[n]} x_i = \frac{n(n+1)}{2} \}$

1) $\mathrm{vert}(\Pi_n) \subset T$. ✓

2) $\mathrm{vert}(\Pi_n) \subset \mathrm{vert}(T)$: $v \in \mathrm{vert}(\Pi_n)$. Let $A_k = \{ i \in [n] \mid v_i \leq k \}$.

$\{v\} = \{ x \mid \sum_{i\in A_k} x_i = \frac{k(k+1)}{2},\ k \in [n] \}$

By 1) we know none of these are cut off!

3) If v is a solution to a system of n equations of the form

$$\sum_{i\in A_k} x_i = \frac{k(k+1)}{2}$$

where $|A_k| = k$ and the cardinalities are pairwise distinct. Let

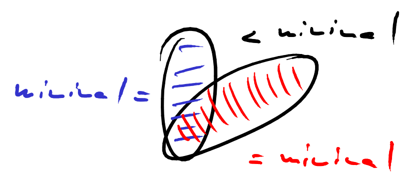
$$A_1 \subset \dots \subset A_j \not\subset A_{j+1} .$$

Suppose $v \in \operatorname{vert}(T)$

$$\underbrace{\sum_{i\in A_{j+1}\cup A_j} v_i}_{\geq \text{minimal}} = \sum_{i\in A_j} v_i + \sum_{i\in A_{j+1}} v_i - \underbrace{\sum_{i\in A_j\cap A_{j+1}} v_i}_{\geq \text{minimal}}$$

$s := |A_{j+1} \cap A_j|$

$$\Rightarrow \quad \frac{(2j+1-s)(2j+2-s)}{2} \leq \frac{j(j+1)}{2} + \frac{(j+1)(j+2)}{2} - \frac{s(s+1)}{2}$$



$$\Rightarrow \quad 2(j-s)^2 + 2(j-s) \leq 1 \quad ↯$$

4) If v is a solution to a system of n equations of the form

$$\sum_{i \in A_k} v_i = \frac{|A_k|(|A_k|+1)}{2}$$

such that $|A_i| = |A_j|$ for some $i \neq j$, then derive a contradiction with similar arguments.

5) 3) + 4) $\Rightarrow$ $\mathrm{vert}(\Pi_n) = \mathrm{vert}(T)$ $\square$.

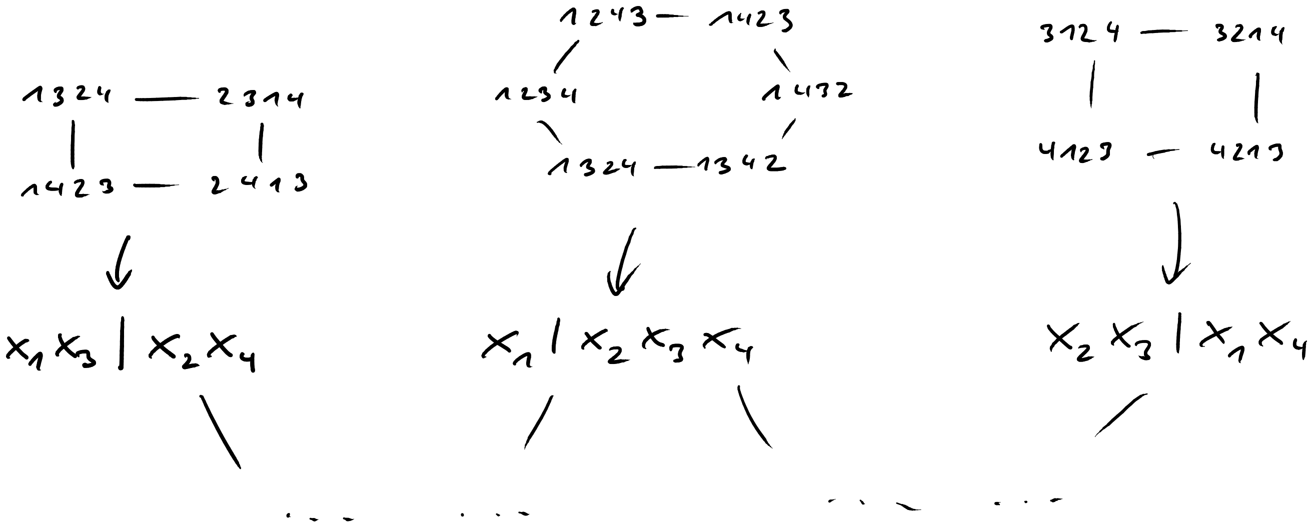
Theorem: The facet-defining inequalities of Π_n are $\sum_{i \in A} x_i \geq \frac{k(k+1)}{2}$ where $A \in \binom{[n]}{k}$, $k \in [n-1]$.

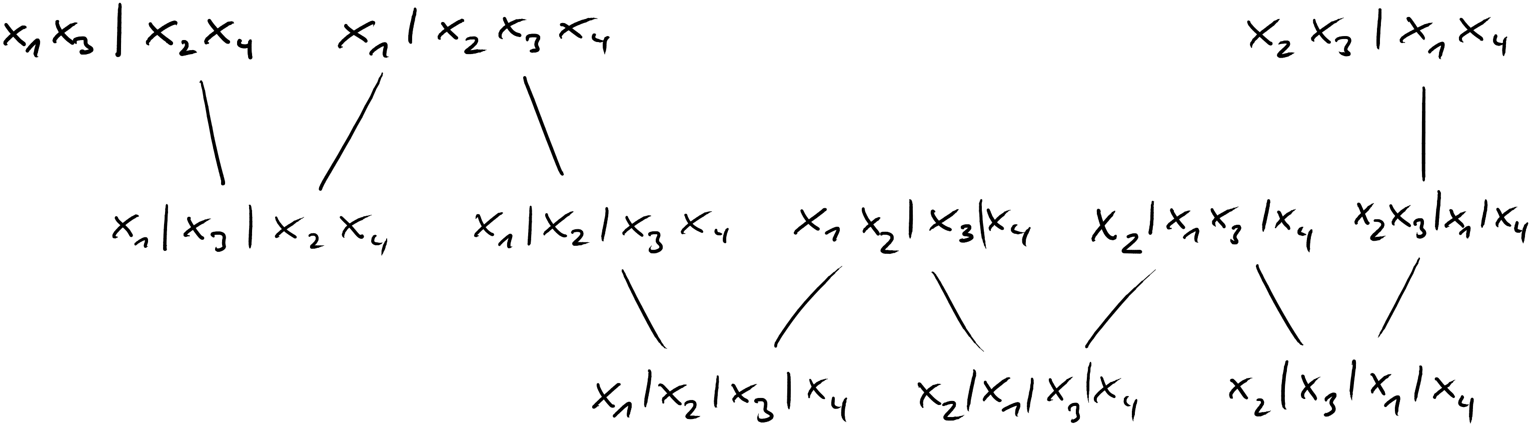
Combinatorial Interpretation Recall $\dim \Pi_n = n-1$

The facets of Π_n correspond to ordered partitions of $[n]$ into 2 non-empty parts.

The vertices of Π_n correspond to ordered partitions of $[n]$ into n non-empty parts (i.e. permutations!)

The k-faces of Π_n correspond to ordered partitions of $[n]$ into $n-k$ non-empty parts.


1324 — 2314
1423 — 2413
$x_1 x_3 \mid x_2 x_4$
1243 — 1423
1234
1432
1324 — 1342
$x_1 \mid x_2 x_3 x_4$
3124 — 3214
4123 — 4213
$x_2 x_3 \mid x_1 x_4$


$x_1x_3 \mid x_2x_4$
$x_1 \mid x_2x_3x_4$
$x_2x_3 \mid x_1x_4$
$x_1 \mid x_3 \mid x_2x_4$
$x_1 \mid x_2 \mid x_3x_4$
$x_1x_2 \mid x_3 \mid x_4$
$x_2 \mid x_1x_3 \mid x_4$
$x_2x_3 \mid x_1 \mid x_4$
$x_1 \mid x_2 \mid x_3 \mid x_4$
$x_2 \mid x_1 \mid x_3 \mid x_4$
$x_2 \mid x_3 \mid x_1 \mid x_4$