Social Choice Theory

- ▷ Preference Aggregation:

  Suppose we know what every individual in society wants. How can we aggregate these individual preferences into a social choice?

  Social Choice Function: $\{\text{possible choices}\}^{\text{Individuals}} \to \{\text{possible choices}\}$

- ▷ Mechanism Design:

  How can we design a voting procedure such that it is in everyone's best interest to actually reveal their true preferences?

- ▷ Power Indices:

  How much power has an individual in a given voting procedure?

▷ Most famous result: <u>Arrow's Theorem</u>

candidates, choices

preference := linear ordering of a finite set C

social choice function:

$$f\left(\left(\text{lin ord of } C\right)_{v \in \text{Voter}}\right) = \text{lin ord of } C$$

<u>Arrow:</u>

There is no "democratic" social choice function!

▷ huge literature of impossibility theorems.

▷ <u>not</u> the subject for today.

Spatial Voting

$x_0, x_1 \in X$ ← policy space

$x_0 \prec_i x_1 \quad :\Longleftrightarrow$ voter $i \in V$ prefers x_1 to x_0

$x_0 \prec x_1 \quad :\Longleftrightarrow$ the majority of voters prefers x_1 to x_0

What if $X \subset \mathbb{R}^n$?

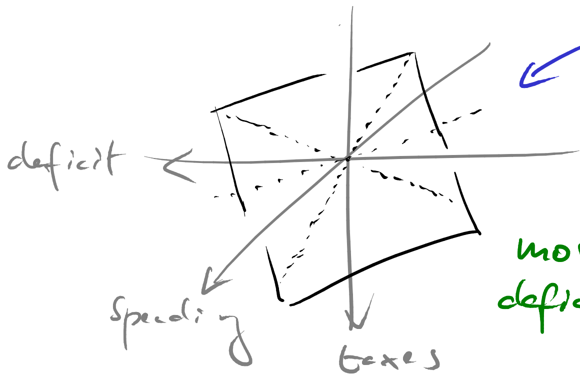
Assumption:

- Each voter $i \in V$ has an ideal policy $p_i \in \mathbb{R}^n$!
- $x_0 \prec_i x_1 \Longleftrightarrow \|p_i - x_0\| < \|p_i - x_1\|$

Euclidean distance

Example: Budget ← (at least) 2-dimensional

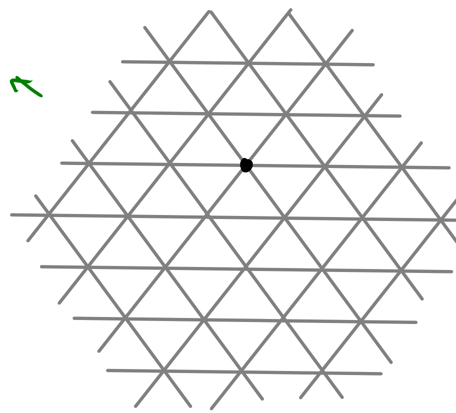
$$\text{spending} - \text{taxes} = \text{deficit}$$



budget decisions are points in this hyperplane

more deficit

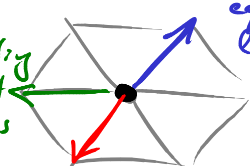
less spending



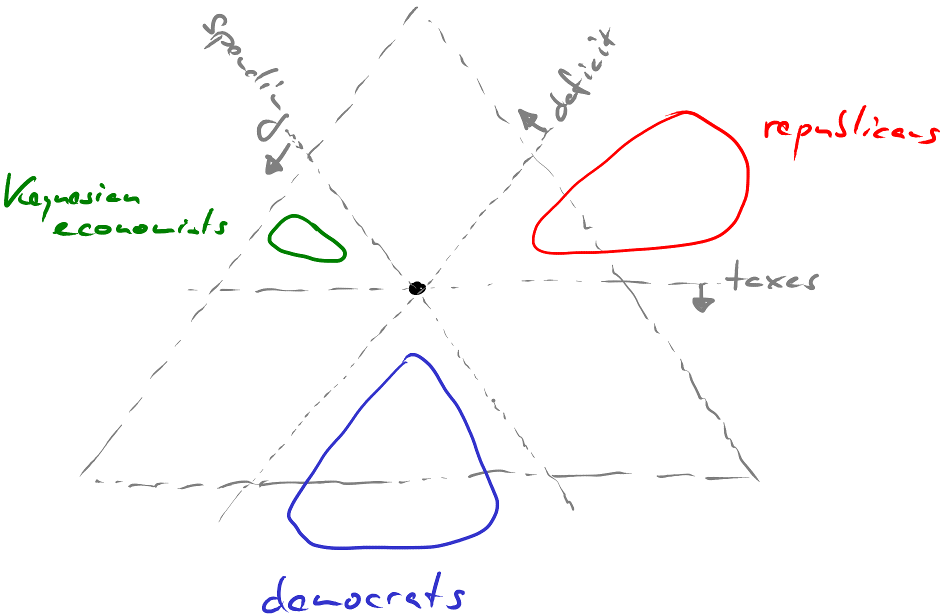
more taxes

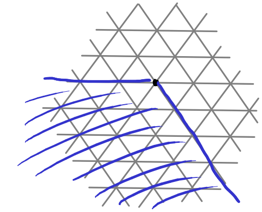
less spending
equal deficit
less taxes

more spending
more deficit
equal taxes



more spending
equal deficit
more taxes





spending and taxes non-negative

Agenda Control

decide between two alternatives:

status quo x_0

x_1 new policy proposed by bill

McKelvey's Theorem

Given $x, y \in \mathbb{R}^n$. Then there exist $x_i \in \mathbb{R}^n$ such that

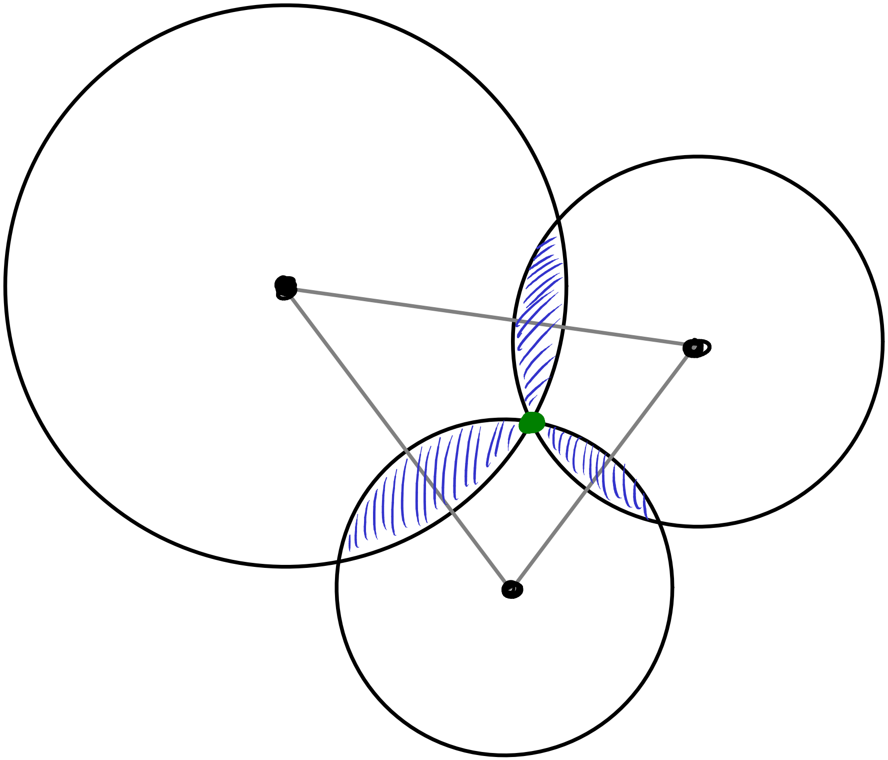
second poll, n-th poll

$$x = x_0 \prec x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_{n-1} \prec x_n = y$$

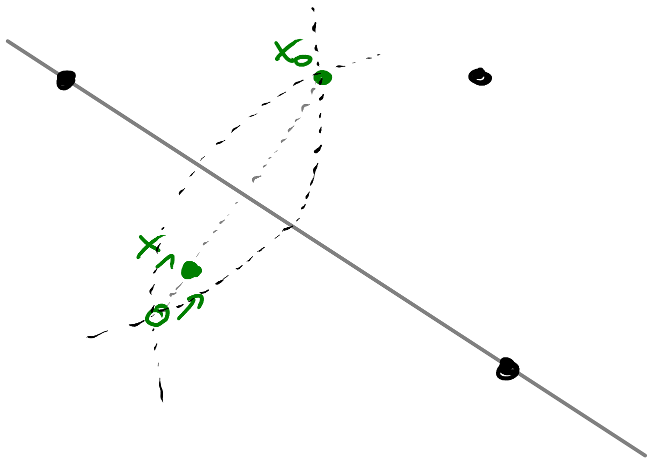
first poll, third poll

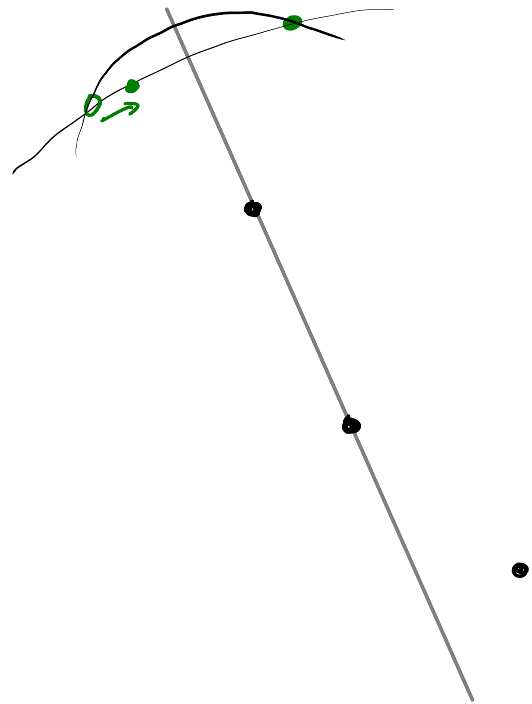
You can get voters to approve any bill, if you control the agenda.

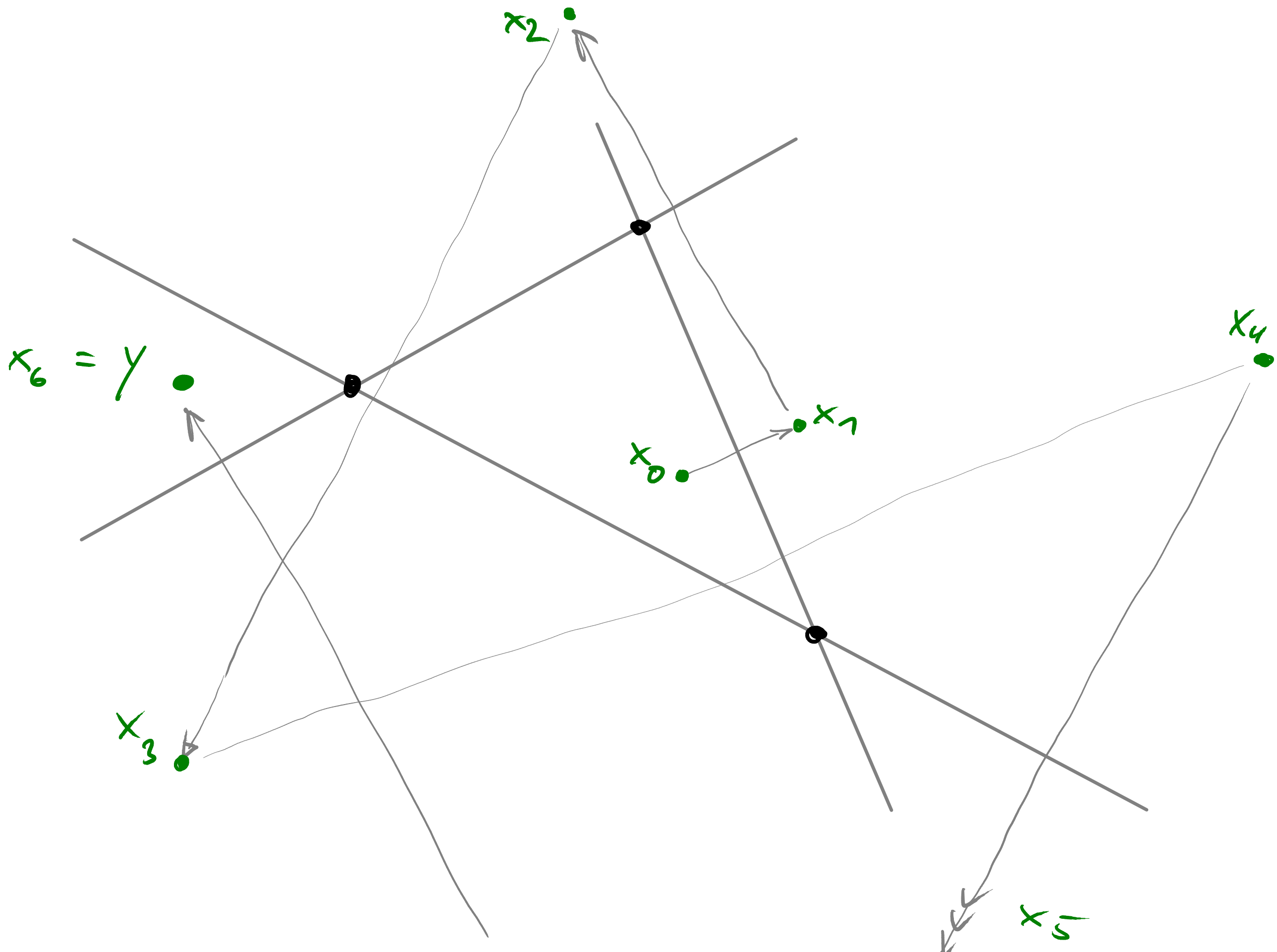
Which x_1 defeat x_0 in a majority vote?



Reflection at one of these lines always works!






x_2
x_4
$x_6 = y$
x_1
x_0
x_3
x_5

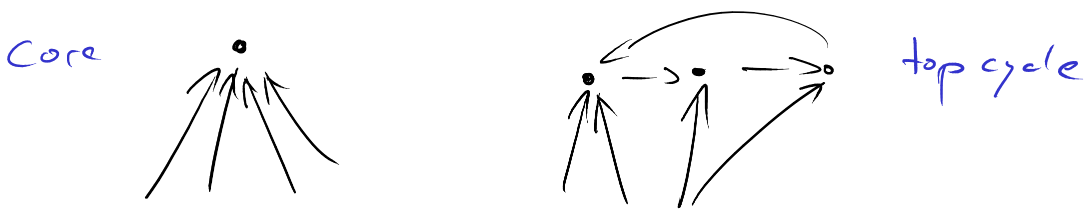
McKelvey's Theorem   dimension ≥ 2, #voters ≥ 3

If the core of the spatial voting game is empty, its top cycle is $\mathbb{R}^n$.

Assume: # voters is odd.

Def: core $= \{ x \mid y \prec x \quad \forall y \in \mathbb{R}^n \}$.

top cycle $= \{ x \mid \forall y \; \exists x_i : y \prec x_0 \prec \ldots \prec x_n \prec x \}$

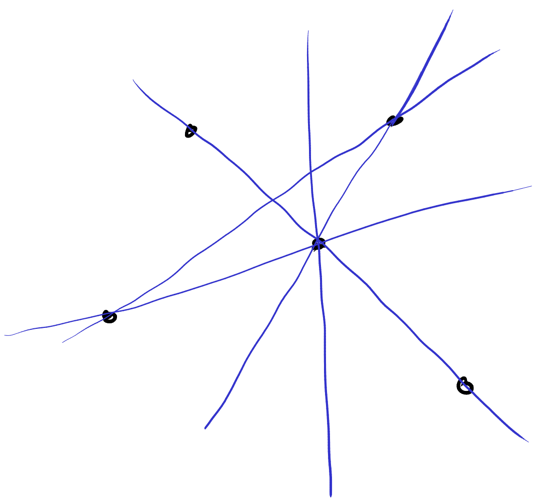


Thm: In general, the core is empty in dim ≥ 2.

Median Hyperplane Let $V = (p_v \mid p_v$ ideal policy of voter $v)$

For every normal a there exists b_a such that

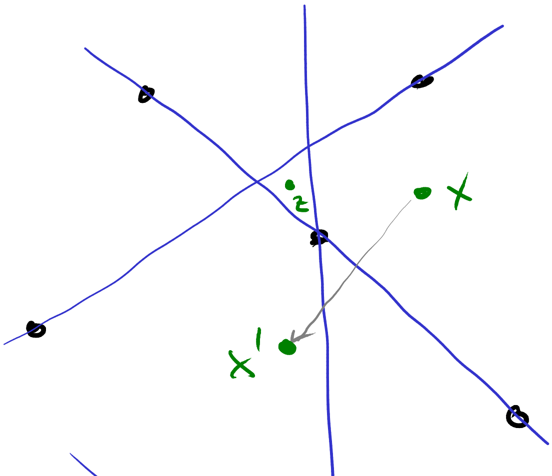
$$\left|H^+_{a,b_a} \cap V\right| \geq \frac{|V|}{2} \leq \left|H^-_{a,b_a} \cap V\right|$$

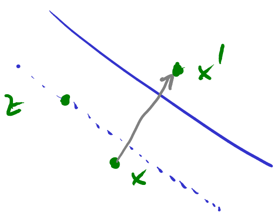


- ▷ #voters is odd $\Rightarrow b_a$ is unique
- ▷ core $= \bigcap_a H_{a,b_a}$

$\ulcorner$ Let $x \notin H_{a,b_a}$. Let $\pi(x)$ be the orth. proj. of x onto H_{a,b_a}. Then $x \prec \pi(x)$. $\lrcorner$

Proof Sketch for McKelvey's Thm





- Pick $d+1$ median hyperplanes
  $$\bigcap_{i=1}^{d+1} H_{a_i, b_{a_i}}$$
- Pick a point z "inside the triangle".
- Show that $\forall x \in \mathbb{R}^d$ there ex i with $x \notin H_{a_i, b_{a_i}}$, s.t. by moving x "across" $H_{a_i, b_{a_i}}$ you get x' s.t. $x \prec x'$ and $\|x - z\| < \|x' - z\|$.

- Construct $x_0 \prec x_1 \prec \dots \prec x_{n-1}$ that gets arbitrarily far away from z.
- If x_{n-1} is "far away enough", going to y will be an improvement for everybody.

$\square$

Generalizations:

If $X \subset \mathbb{R}^n$ is connected, and voters have a continuous utility function, then, in general, the top cycle of the majority rule is always going to be all of X.

"Regardless of other voter's preferences, any one voter with complete information of the other voter's preferences, control of the agenda, and the ability to cast his own vote as he chooses can always construct majority paths to get anywhere in space."

Strong Point

$$\text{defeat}(x) = \text{Vol}\{y \mid y \succ x\}$$

$x = \arg\min_x \text{defeat}(x)$ is called the strong point

<u>Thm</u> x exists and is unique

<u>Thm</u> In dimension 2:

$x = \sum \lambda_i v_i$ convex combination of voter's ideal policies v_i

λ_i = <u>spatial power index</u> of voter i

Roughly: centrist voters have more power.