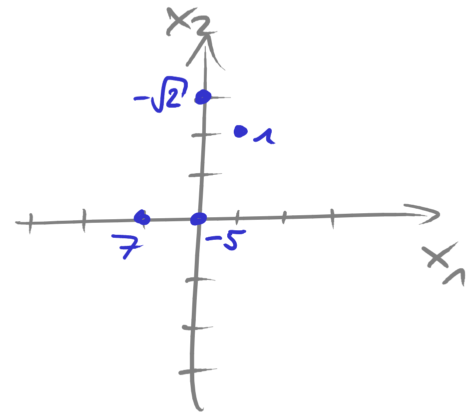
(Laurent) polynomial $p(x) \in \mathbb{R}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$

$$x_1 x_2^2 + 7x_1^{-1} - \sqrt{2}\, x_2^3 - 5$$



finite multiset of lattice points in $\mathbb{Z}^d$.

$$\left\{ \begin{pmatrix}1\\2\end{pmatrix} : 1, \begin{pmatrix}-1\\0\end{pmatrix} : 7, \begin{pmatrix}0\\3\end{pmatrix} : -\sqrt{2}, \begin{pmatrix}0\\0\end{pmatrix} : -5 \right\}$$

function $\mathbb{Z}^d \to \mathbb{R}$ with finite support.

$f(1,2) = 1$ $\quad f(-1,0) = 7$ $\quad f(0,3) = -\sqrt{2}$ $\quad f(0,0) = -5$ $\quad f(x,y) = 0$ otherwise

$$x^a = x_1^{a_1} \cdot \ldots \cdot x_d^{a_d} \quad \text{for} \quad a \in \mathbb{Z}^d$$

sum of polynomials $\Longleftrightarrow$ (disjoint) union of multisets

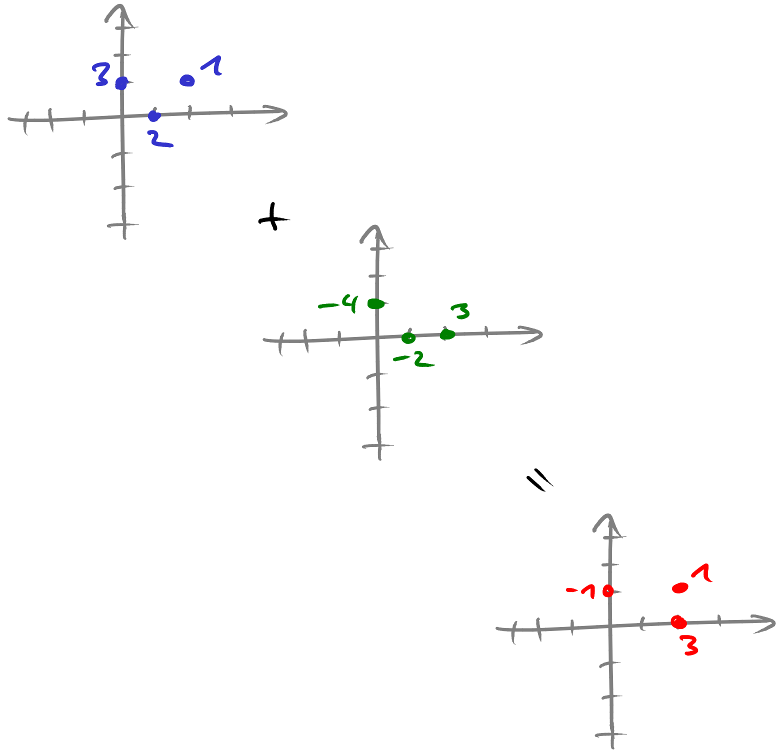
$$x_1^2 x_2 + 3x_2 + 2x_1$$

$$+$$

$$-4x_2 + 3x_1^2 - 2x_1$$

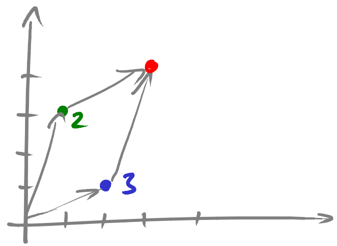
$$=$$

$$x_1^2 x_2 - x_2 + 3x_1^2$$



product of monomials $\longleftrightarrow$ sum of vectors,
product of coefficients

$$3x_1^2 x_2^1 \cdot 2x_1 \cdot x_2^3 = 6 \cdot x_1^{2+1} \cdot x_2^{1+3}$$

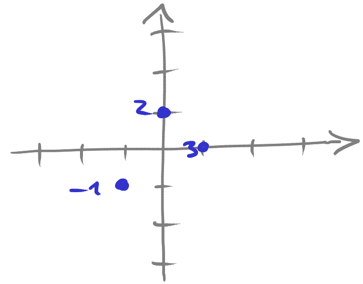


How do monomial ideals look, then?

$$\langle x_1^1 x_2^4,\ x_1^3 x_2^2,\ x_1^1 x_2^5 \rangle \subset \mathbb{R}[x_1, x_2]$$

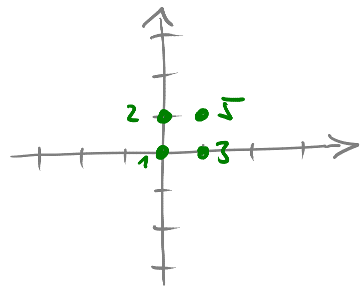
product of polynomials $\Leftrightarrow$ Minkowski sum of multisets.

$$-x_1^{-1}x_2^{-1} + 3x_1 + 2x_2$$



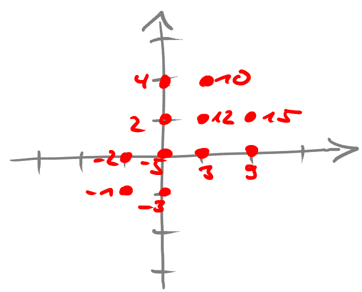
$$\cdot$$

$$1 + 3x_1 + 2x_2 + 5x_1x_2$$



$$=$$

$$-x_1^{-1}x_2^{-1} - 3x_2^{-1} - 2x_1^{-1} - 5$$
$$+ 3x_1 + 9x_1^2 + 6x_1x_2 + 15x_1^2x_2$$
$$+ 2x_2 + 6x_1x_2 + 4x_2^2 + 10x_1x_2^2$$



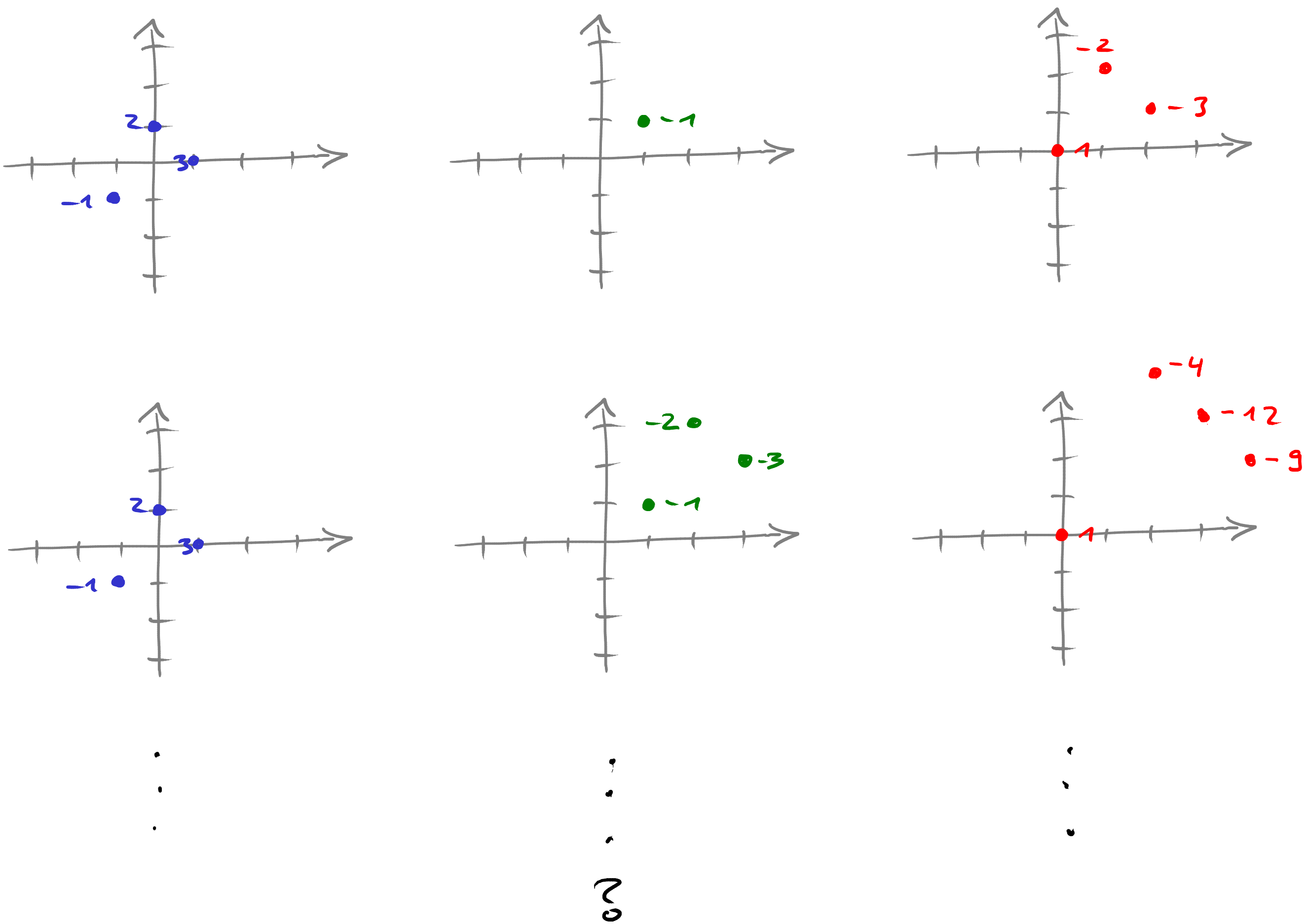
product of polynomials $\Leftrightarrow$ Minkowski sum of multisets.

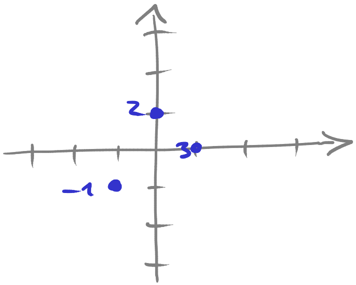
$$\left(\sum_{a\in A} \alpha_a x^a\right)\cdot\left(\sum_{b\in B} \beta_b x^b\right)$$

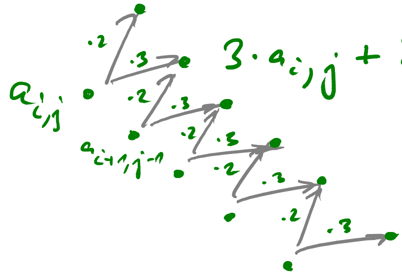
$$= \sum_{a\in A}\sum_{b\in B} \alpha_a\cdot\beta_b\, x^{a+b}$$

$$= \sum_{c\in A+B}\left(\sum_{\substack{a\in A\\ b\in B\\ a+b=c}} \alpha_a\cdot\beta_b\right) x^c$$

multiplicative inverse $\Longleftrightarrow$ "inverse" Minkowski sum ?

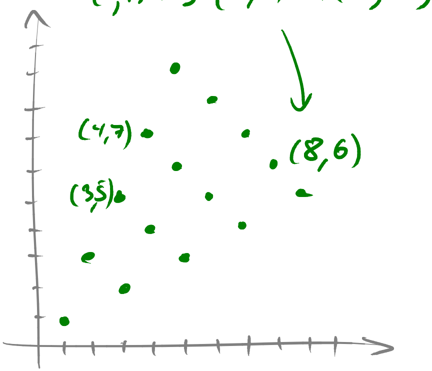




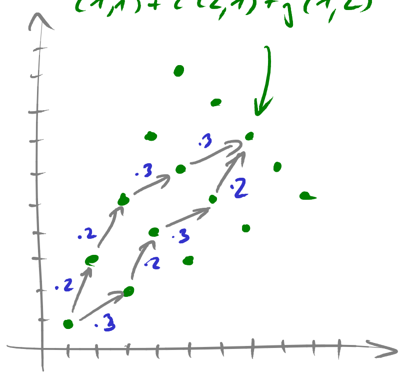


$$3\cdot a_{i,j} + 2\cdot a_{i+1,j-1}$$

$(1,1)+3\cdot(2,1)+1(1,2)$



$(1,1)+i\,(2,1)+j\,(1,2)$



$$\sum_{i,j\ge 0} a_{i,j}\, x_1^{1+2i+j}\, x_2^{1+i+2j}$$

$$a_{i,j} = \binom{i+j}{i} 3^i 2^j$$

multiplicative inverse of polynomial

$$\frac{1}{-x_1^{-1}x_2^{-1}+3x_1+2x_2}$$

$\Updownarrow$

"Minkowski inverse" of finite multiset

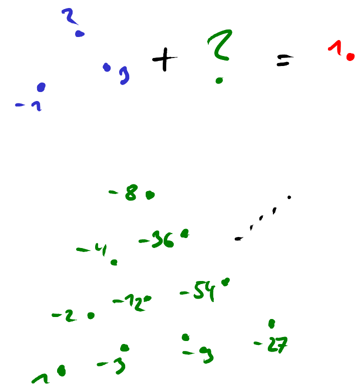
$\Updownarrow$

infinite "periodic" lattice point set with weights given by linear recurrence

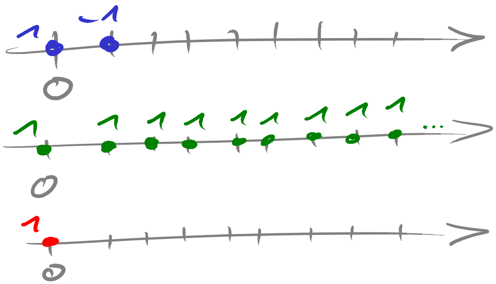
$\Updownarrow$

formal power series

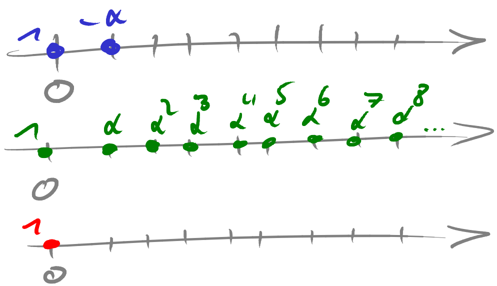
$$\sum_{i,j\geq 0}\binom{i+j}{i}3^i2^j\,x_1^{1+2i+j}\,x_2^{1+i+2j}$$



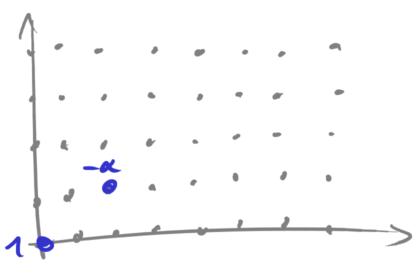
"easy" inverses

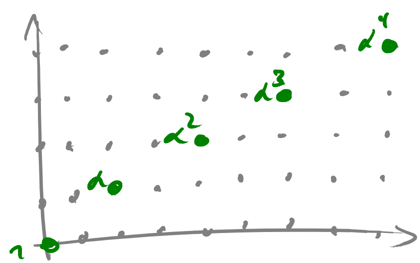


$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i$$



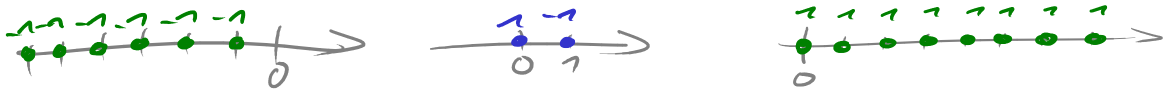
$$\frac{1}{1-\alpha z} = \sum_{i=0}^{\infty} \alpha^i z^i$$





$$\frac{1}{1-\alpha z^a} = \sum_{i=0}^{\infty} \alpha_i \, z^{a\cdot i}$$

"Minkowski inverse" not uniquely determined!



$$\sum_{i\geq 1}^{\infty} -z^{-i} = \frac{1}{1-z} = \sum_{i\geq 0}^{\infty} z^{i}$$

So:

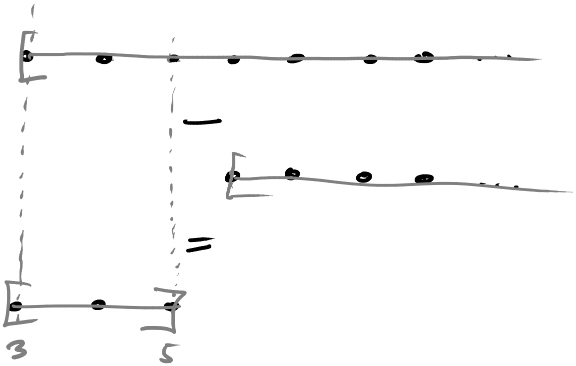
$$\sum_{i\in\mathbb{Z}} z^{i} = \sum_{i\geq 1}^{\infty} z^{-i} + \sum_{i\geq 0}^{\infty} z^{i}$$

$$= -\left(\sum_{i\geq 1}^{\infty} -z^{-i}\right) + \left(\sum_{i\geq 0}^{\infty} z^{i}\right)$$

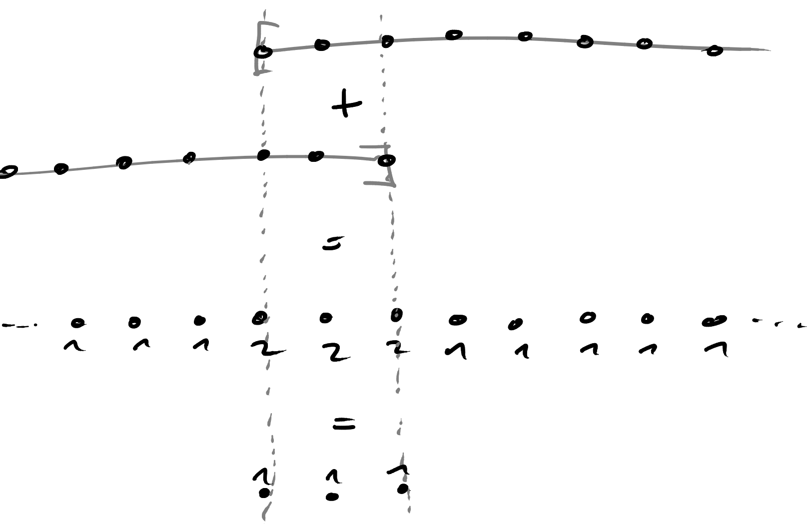
$$= -\frac{1}{1-z} + \frac{1}{1-z} = 0$$

$\Rightarrow$ Consider equivalence classes up to lines in $\mathbb{Z}^d$.

two ways of representing intervals



$$\frac{z^3}{1-z} + \frac{z^6}{1-z} = \frac{z^3 - z^6}{1-z}$$

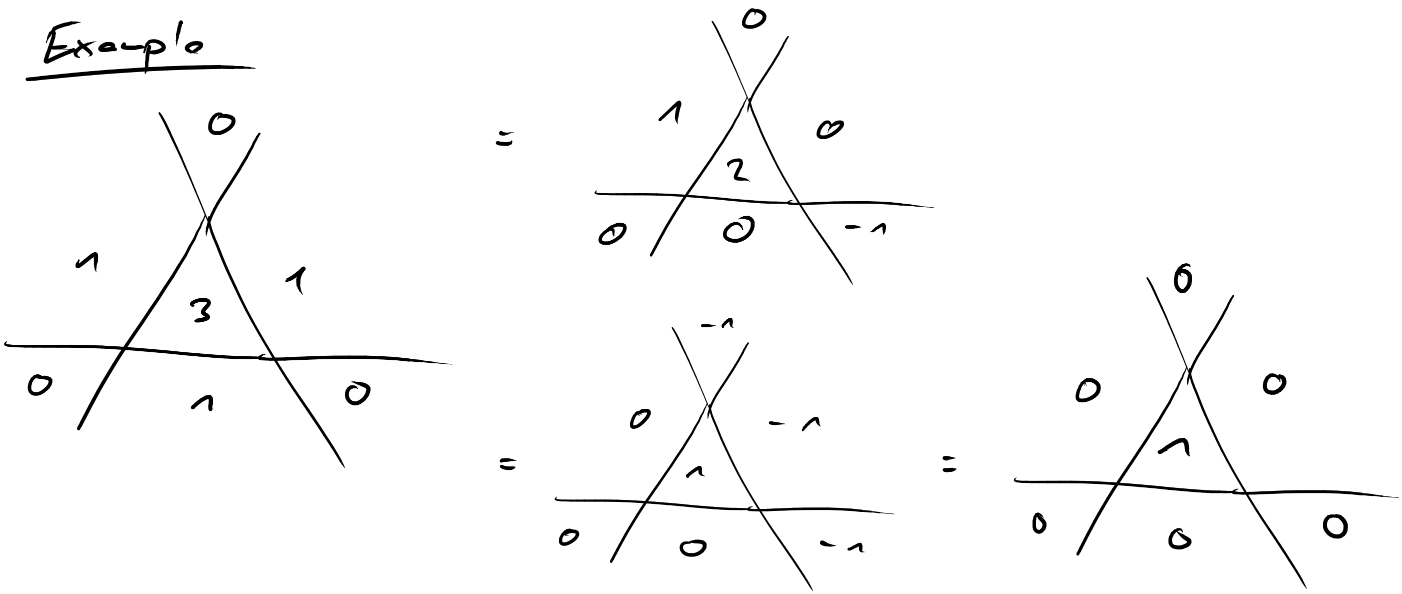


$$\frac{z^3}{1-z} + \frac{z^6}{1-z} = \frac{z^3 + z^6}{1-z}$$

Brion's Theorem Let P be a polytope. For every vertex $v \in V(P)$, let $C_v = \mathrm{cone}(v_1,\dots,v_k)+v$ be the cone generated by all "edge directions" at v. Let $\sigma_v(z)$ be the associated formal power series and let $\sigma(z) = \sum_{x \in P \cap \mathbb{Z}^d} z^x$. Then

$$\sigma(z) = \sum_{v \in V(P)} \sigma_v(z).$$

Example



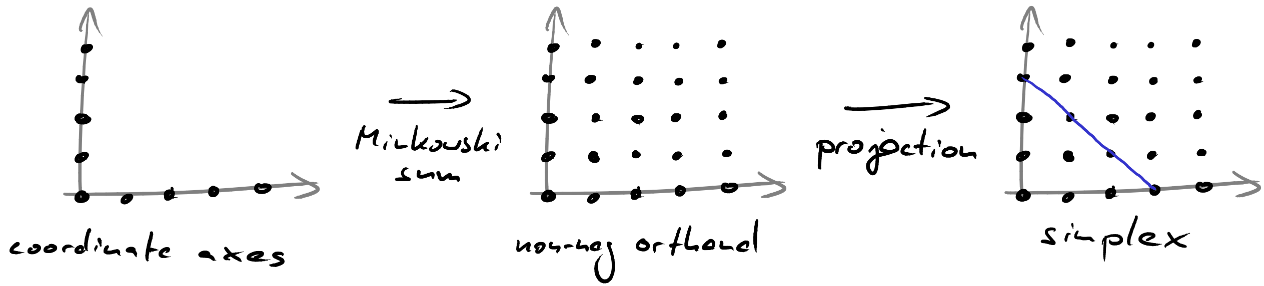
$$\frac{1}{(1-z)^d} = \prod_{i=1}^{d} \frac{1}{1-z} = \underset{\text{Minkowski}}{\sum_{i=1}^{d}} \left(\underset{\bullet}{1}\ \underset{\bullet}{1}\ \underset{\bullet}{1}\ \underset{\bullet}{1}\ \underset{\bullet}{1}\ \underset{\bullet}{1}\ \ldots \right)$$

$$\frac{1}{1-z^2} = (1\ 2\ 3\ 4\ 5\ 6\ \ldots)$$

$$\frac{1}{1-z^3} = (1\ 3\ 6\ 10\ 15\ 21\ \ldots)$$

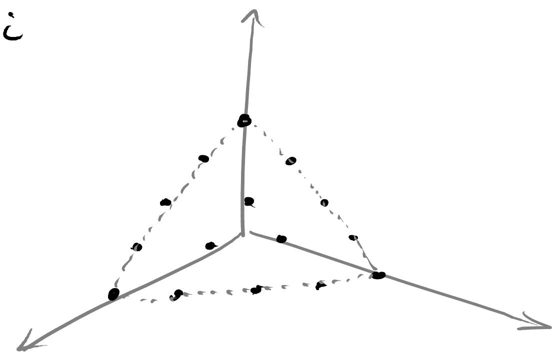
$$\frac{1}{1-z^4} = (1\ 4\ 10\ 20\ 35\ 56\ \ldots)$$

$$f(z_1,\dots,z_d) = \prod_{i=1}^{d} \frac{1}{(1-z_i)} = \sum_{j_1\geq 0}\dots\sum_{j_d\geq 0} z_1^{j_1}\cdots z_d^{j_d} = \sum_{v\in\mathbb{Z}^d_{\geq 0}} z^v$$



$$\frac{1}{(1-z)^d} = f(z,\dots,z) = \sum_{i\geq 0}\Big(\sum_{\substack{j_1,\dots,j_d\geq 0\\ j_1+\dots+j_d=i}} 1\Big) z^i$$

$$\frac{1}{(1-z)^d} = \sum_{i\geq 0}\binom{i+d-1}{d-1} z^i$$



$$\frac{1}{(1-z)^{d+1}} = \sum_{k\geq 0} \binom{k+d}{d} z^k \quad \Rightarrow$$

$$\frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^{d+1}} = \sum_{i=0}^{d} h_i \frac{z^i}{(1-z)^{d+1}} = \sum_{i=0}^{d} h_i \sum_{k\geq 0} \binom{k+d}{d} z^{k+i}$$

$$= \sum_{i=0}^{d} h_i \sum_{k\geq i} \binom{k+d-i}{d} z^k = \sum_{i=0}^{d} h_i \sum_{k\geq 0} \binom{k+d-i}{d} z^k$$

$$= \sum_{k\geq 0} \left(\sum_{i=0}^{d} h_i \binom{k+d-i}{d} \right) z^k$$

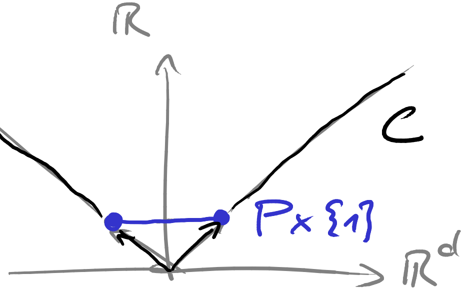
<u>Lemma</u> Let $\frac{p(z)}{(1-z)^{d+1}} = \sum_{k\geq 0} q(k) z^k$. Then

p is a polynomial of deg $\leq d$ $\Leftrightarrow$ q is a polynomial of deg $\leq d$.

coeffs wrt. $(x^i)_{0\leq i\leq d}$ = coeffs wrt. $\binom{k+d-i}{d}_{0\leq i\leq d}$.

P lattice simplex, <u>not</u> unimodular, $V(P) = v_1, \dots, v_{d+1}$

$$C = \operatorname{cone}\left(\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_{d+1} \\ 1 \end{pmatrix}\right) \subset \mathbb{R}^{d+1}$$



$$L_P(k) = \# C \cap \{x \mid x_{d+1} = k\} \cap \mathbb{Z}^{d+1}$$

Partition lattice points in C into translates of $\displaystyle\prod_{i=1}^{d+1} \frac{1}{\left(1 - z^{\binom{v_i}{1}}\right)}$.

$$\Rightarrow \sum_{x \in \mathbb{Z}^{d+1} \cap C} z^x = p(z) \cdot \prod_{i=1}^{d+1} \frac{1}{\left(1 - z^{\binom{v_i}{1}}\right)}.$$

for a polynomial p with max exponent of $z_{d+1} \le d$

$$\sum_{x \in \mathbb{Z}^{d+1} \cap C} z^x = p(z) \cdot \prod_{i=1}^{d+1} \frac{1}{(1 - z^{(v_i)})}$$

$d+1$ variables $z_1, \dots, z_{d+1}$

How many terms are there with $z_{d+1}^{x_{d+1}} = z_{d+1}^k$?

$$p(1, \dots, 1, z_{d+1}) = \sum_{i=0}^{d} h_i z_{d+1}^i$$

$$\sum_{k \geq 0} L_P(k) z_{d+1}^k = p(1, \dots, 1, z_{d+1}) \cdot \frac{1}{(1 - z_{d+1})^{d+1}}$$

$$= \frac{\sum_{i=0}^{d} h_i z_{d+1}^i}{(1 - z_{d+1})^{d+1}} = \sum_{k \geq 0} \left(\sum_{i=0}^{d} h_i \binom{k+d-i}{d} \right) z_{d+1}^k$$

<u>Ehrhart's Theorem</u> $L_P(k) = \left(\sum_{i=0}^{d} h_i \binom{k+d-i}{d} \right)$ where

h_i = # lattice points in the fundamental parallelepiped of C at level i.