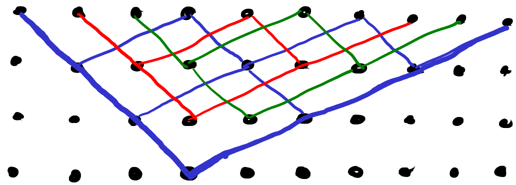
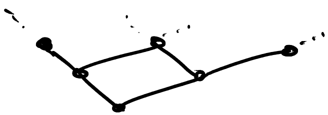
$P = [-1, 2] \longrightarrow P \times \{1\} \subset \mathbb{R}^2$



$$\frac{1}{(1-z)^2} + \frac{z}{(1-z)^2} + \frac{z}{(1-z)^2} = \frac{1+2z}{(1-z)^2}$$



$$= \frac{1}{(1 - z_1^{-1} z_2^{1})(1 - z_1^{2} z_2^{1})} \xrightarrow[z_2 \to z]{z_1 \to 1} \frac{1}{(1-z)^2}$$

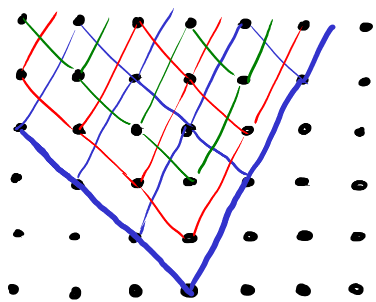
If P is a simplex of dimension d:

$$\sum_{k \geq 0} L_P(k) z^k = \frac{\sum_{i=0}^{d} h_i z^i}{(1-z)^{d+1}} = \sum_{k \geq 0} \left(\sum_{i=0}^{d} h_i \binom{k+d-i}{d} \right) z^k$$

where h_i = # lattice points in fundamental parallelepiped at height k.

Same thing works for rational polytopes!

$$P = [-1, \tfrac{1}{2}] \longrightarrow P \times \{1\} \subset \mathbb{R}^2$$



$$\frac{1}{(1-z)(1-z^2)} + \frac{z}{(1-z)(1-z^2)} + \frac{z^2}{(1-z)(1-z^2)}$$

$$= \frac{1+z+z^2}{(1-z)(1-z^2)}$$

$$\sum_{k \geq 0} L_P(k)\, z^k = \frac{1+z+z^2}{(1-z)(1-z^2)} = \sum_{k \geq 0} ?\; z^k$$

$$
\begin{array}{cccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
= & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5
\end{array}
$$

polynomial part periodic part

$$\frac{1}{(1-z)} \cdot \frac{1}{(1-z^2)} = \sum_{k \geq 0} \left(\frac{1}{2}(k+1) + \left\{ \frac{k+1}{2} \right\} \right) z^k$$

$$= \frac{\frac{1}{2}}{(1-z)^2} + \frac{\frac{1}{2}}{(1-z^2)}$$

$$
\begin{array}{ccccccccc}
\frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} & 4 & \frac{9}{2}
\end{array}
\quad + \quad
\begin{array}{ccccccccc}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{array}
$$

Lemma $$\frac{p(z)}{(1-z)^{d+1}} = \sum_{k\geq 0} q(z) z^k$$

$p(z)$ is a polynomial of degree $\leq d$

$\Longleftrightarrow$ $q(z)$ is a polynomial of degree $\leq d$

Lemma $$\frac{p(z)}{(1-z)^{d+1}} = \sum_{k\geq 0} q(z) z^k$$

$p(z)$ is a polynomial of degree $\leq d$

$\Longleftrightarrow$ $q(z)$ is a periodic function with period $d+1$

$$\frac{1}{(1-z)} \cdot \frac{1}{(1-z^2)} = \frac{\frac{1}{2}}{(1-z)^2} + \frac{\frac{1}{2}}{(1-z^2)}$$

Tools for finding such decompositions in general:

$$\frac{1}{(1-z)}\cdot\frac{1}{(1-z^2)} = \frac{\frac{1}{2}}{(1-z)^2} + \frac{\frac{1}{2}}{(1-z^2)}$$

Factorization

$\forall p(z)$ w/ $\deg(p)=d$ $\exists\, d_i \in \mathbb{N},\ \alpha_i \in \mathbb{C}$:

$$p(z) = \alpha_0 \cdot \prod_{i=1}^{n} (1-\alpha_i z)^{d_i}$$

Polynomial Division

$\forall a(z), b(z)$ $\exists\, q(z), r(z)$ w/ $\deg(r) < \deg(b)$:

$$a(z) = q(z)\cdot b(z) + r(z)$$

Partial Fraction Decomposition

$\forall p(z)$ $\exists\, q_i(z)$

$$\frac{p(z)}{\prod_{i=1}^{n}(1-\alpha_i z)^{d_i}} = \sum_{i=1}^{n} \frac{q_i(z)}{(1-\alpha_i z)^{d_i}}$$

How to compute Ehrhart functions?

A) Count lattice points in the first dim P dilates of P.

B) Find a unimodular triangulation of P and
- ▷ count its i-faces or
- ▷ construct a shelling and compute its h-vector.

C) Find a triangulation of P and for each simplex σ in that triangulation count the number of lattice points in the fundamental parallelepiped of the cone over σ.

D) Set up the generating function for P and compute its coefficients using partial fractions decomposition.

From H-description to generating function.

$$L_P(k) = \#\{x \in \mathbb{Z}^d \mid x \geq 0 : Ax = kb\}$$

let $m :=$ # rows of $A = (a_1, \dots, a_d)$

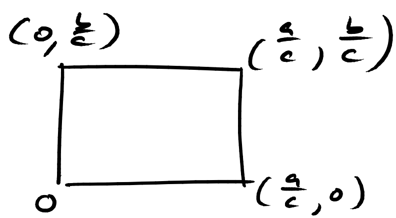
col vector

$$L_P(k) = [z_1^{kb_1} \cdots z_m^{kb_m}] \prod_{i=1}^{d} \sum_{x_i \geq 0} z^{a_i x}$$

coefficient of this monomial in the generating function

$$= [z^{kb}] \prod_{i=1}^{d} \frac{1}{(1 - z^{a_i})}$$

Example $P =$



$$kP = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0, \; x \leq \frac{ak}{c}, \; y \leq \frac{bk}{c}\}$$