Game Theory

Fourier-Motzkin Elimination

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Farkas Lemma

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LP-Duality $\iff$ Minimax-Theorem in Game Theory

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Complementary Slackness

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Max-Flow Min-Cut

Simplex Algorithm

Zero-Sum Two-Person Games

Two players Max, Mini. Finite strategy sets S_{Max}, S_{Mini}.

Rules: Both players simultaneously pick strategies $i \in S_{Max}$, $j \in S_{Mini}$, without knowledge of the other's choice.

Max receives payoff a_{ij}, Mini receives payoff b_{ij}.

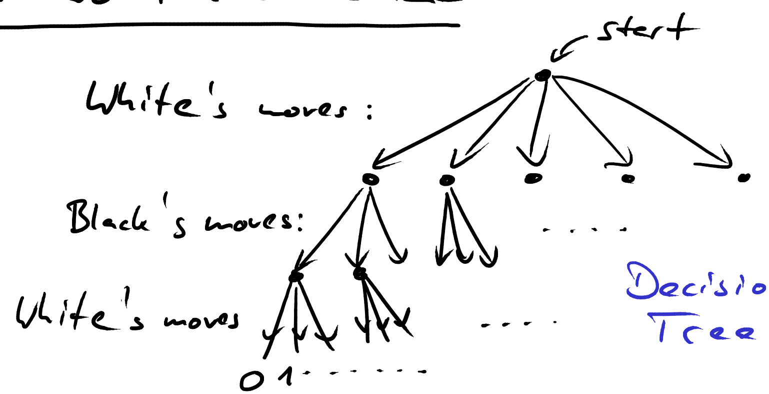
Zero Sum Condition $\quad a_{ij} + b_{ij} = 0 \quad \Leftrightarrow \quad b_{ij} = -a_{ij}$

Payoff Matrix $\quad K(i,j) = A = (a_{ij})_{i \in S_{Max},\, j \in S_{Mini}}$

Max wants to maximize a_{ij}, Mini wants to minimize.

Chess is a Zero-Sum Two Person Game!

• = game state



Decision Tree

this is a game with complete information!

strategy $i \in S_{White}$: $\quad i : \left\{ \begin{array}{l} \text{game states} \\ \text{in which it is} \\ \text{White's turn} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{allowed moves} \\ \text{in that state} \end{array} \right\}$

$$A = (a_{ij}) \qquad a_{ij} = \begin{cases} 1 & \text{White wins} \\ 0 & \text{tie} \\ -1 & \text{Black wins} \end{cases}$$

if White plays i and Black plays j

An example where simultaneous choice matters.

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Max takes a coin in one hand.

Mini chooses one of Max's hands.

- ▷ If Mini finds the coin, Mini wins.
- ▷ Otherwise, Max wins.

Q: How to analyse Two-Person Zero-Sum games?

Approach 1: What does it mean that both players choose optimal strategies?

A Nash-equilibrium or saddle point of a game is a pair of strategies $x^* \in S_{Max}$, $y^* \in S_{Min}$ such that

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y) \quad \forall x, y$$

$$\begin{array}{cl} \begin{pmatrix} & & \leq & & \\ & & \leq & & \\ \geq & \geq & \square & \geq & \geq \\ & & \leq & & \\ & & \leq & & \end{pmatrix} & \text{Max} \\ \text{Mini} & \end{array}$$

Examples

Nash equilibrium

$$\begin{array}{cl} \begin{pmatrix} \boxed{0} & 1 \\ -1 & 0 \end{pmatrix} & \text{Max} \\ \text{Mini} & \end{array}$$

no Nash equilibrium!

$$\begin{array}{cl} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} & \text{Max} \\ \text{Mini} & \end{array}$$

Approach 2: Worst case analysis.

Max does not make any assumptions about Mini's behavior.

For any $x \in S_{Max}$

$$\min_y K(x,y)$$

is the worst-case payoff.

The best Max can do to maximize the worst-case outcome:

$$\max_x \min_y K(x,y).$$

Mini does not make any assumptions about Max's behavior.

For any $y \in S_{Mini}$

$$\max_x K(x,y)$$

is the worst-case payoff.

The best Mini can do to minimize the worst-case outcome:

$$\min_y \max_x K(x,y).$$

$$\max_i \min_j a_{ij} \qquad\qquad \min_j \max_i a_{ij} \qquad\qquad \text{Nash} \; \left(\geq \overset{\leq}{\underset{\leq}{\square}} \geq \right)$$

$$\begin{pmatrix} 7 & 2 & 6 & 3 & \boxed{0} \\ 1 & 1 & 2 & \boxed{0} & 1 \\ 5 & 4 & 3 & \boxed{1} & 2 \\ 2 & \boxed{1} & 3 & 2 & 5 \\ 3 & \boxed{0} & 4 & 5 & 4 \end{pmatrix} \quad 1 < 4 \quad \begin{pmatrix} \boxed{7} & 2 & \boxed{6} & 3 & 0 \\ 1 & 1 & 2 & 0 & 1 \\ 5 & \boxed{4} & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & \boxed{5} \\ 3 & 0 & 4 & \boxed{5} & 4 \end{pmatrix}$$

no!

$$\begin{pmatrix} \boxed{0} & 1 \\ \boxed{-1} & 0 \end{pmatrix} \quad 0 = 0 \quad \begin{pmatrix} \boxed{0} & \boxed{1} \\ -1 & 0 \end{pmatrix}$$

yes!

$$\begin{pmatrix} \boxed{-1} & 1 \\ 1 & \boxed{-1} \end{pmatrix} \quad -1 < 1 \quad \begin{pmatrix} -1 & \boxed{1} \\ \boxed{1} & -1 \end{pmatrix}$$

no!

<u>Thm:</u> $\max_x \min_y K(x,y) \leq \min_y \max_x K(x,y)$

<u>Thm:</u> Equality holds $\Leftrightarrow$ the game has a Nash equilibrium.

Max's worst-case optimization | Mini's worst-case optimization

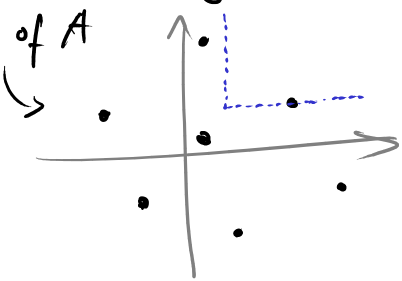
$$\overbrace{\max_x \underbrace{\min_y K(x,y)}_{\substack{\text{given } x\in S_{Max} \\ \text{Mini chooses best } y\in S_{Mini}}}}^{\text{Mini knows Max's strategy}} \le \overbrace{\min_y \underbrace{\max_x K(x,y)}_{\substack{\text{given } y\in S_{Mini} \\ \text{Max chooses best } x\in S_{Max}}}}^{\text{Max knows Mini's strategy}}$$

<u>Intuition:</u> We have equality when both players have complete information.

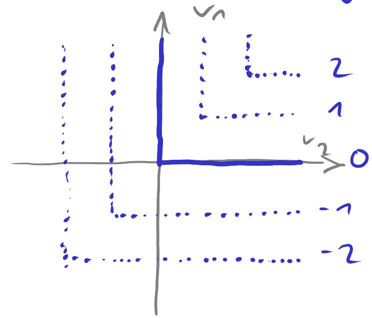
Max's POV:

$$\max_i \min_j a_{ij}$$

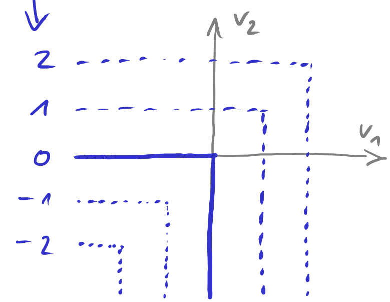
rows of A



$\min(v_1, v_2)$



$\max(v_1, v_2)$

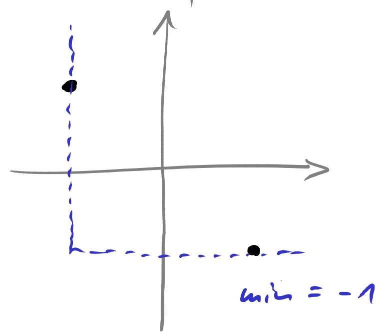


Example

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

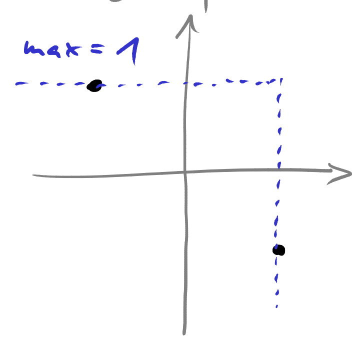
When you play this game in practice, you randomize!

row space



$$\max_i \min_j a_{ij}$$

col space



$$\min_j \max_i a_{ij}$$

Mixed Extension of a Game

A *mixed strategy* for player P is a probability measure λ on S_P, in other words a vector $x \in \Delta^{S_P}$ where

$$\Delta^{S_P} = \{ x \in \mathbb{R}^{S_P} \mid x_i \geq 0 ,\ \sum x_i = 1 \}$$ is the *probability simplex*.

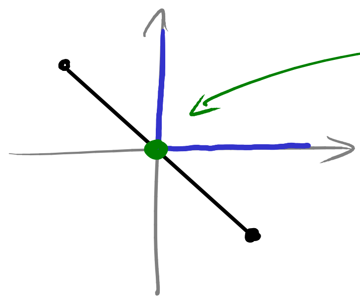
Each $i \in S_P$ is called a *pure strategy*.

If Max plays $x \in \Delta^{S_{Max}}$ and Mini plays $y \in \Delta^{S_{Mini}}$ then the *payoff* is the *expected payoff*

$$k(x,y) = \sum_{i \in S_{Max}} \sum_{j \in S_{Mini}} x_i y_j a_{ij}$$

when Max chooses strategy $i \in S_{Max}$ with probability x_i, and Mini chooses strategy $j \in S_{Mini}$ with probability y_j.

A mixed strategy $x \in \Delta^{S_{Max}}$ corresponds to a convex combination of pure strategies $\sum x_i e_i$!



Assume Mini plays a pure strategy.

$$\max_{x \in \Delta^{S_{Max}}} \min_{j} \sum_i x_i a_{ij} = 0$$

$$\| \quad \leftarrow \text{Lemma}$$

$$\max_{x \in \Delta^{S_{Max}}} \min_{y \in \Delta^{S_{Mini}}} \sum_{i,j} x_i y_j a_{ij}$$

▷ If Max plays $(\frac{1}{2}, \frac{1}{2})$ he will win half the time, no matter which strategy Mini uses! No other $x \in \Delta^{S_{Max}}$ can do better.

▷ $$\max_x \min_y K(x,y) = \min_y \max_x K(x,y)$$

in the extended game!

Consider the mixed extension of <u>any</u> two person zero sum game.

Minimax Theorem

There exist mixed strategies $x^* \in \Delta^{S_{max}}$, $y^* \in \Delta^{S_{mini}}$ such that

$$K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y) \qquad \forall x, y.$$

Equivalently:

$$\max_x \min_y K(x,y) = \min_y \max_x K(x,y).$$

<u>Proof:</u> LP Duality!

$$\max_{x \in \Delta^{S_{max}}} \min_{y \in \Delta^{S_{min}}} K(x,y)$$

$$= \max_{x \in \Delta^{S_{max}}} \min_{j} \sum_i x_i a_{ij}$$

$$= \max \left\{ c \in \mathbb{R} \mid c = \min_j \sum_i x_i a_{ij},\ x \in \Delta^{S_{max}} \right\}$$

$$= \max \left\{ c \in \mathbb{R} \mid c \leq \min_j \sum_i x_i a_{ij},\ x \in \Delta^{S_{max}} \right\}$$

$$= \max \left\{ c \in \mathbb{R} \mid \forall j : c \leq \sum_i x_i a_{ij},\ x \in \Delta^{S_{max}} \right\}$$

$$\begin{aligned} = \ \max \quad & c \\ \text{subto} \quad & c \leq \sum_i x_i a_{ij} && \forall j \\ & 1 = \sum_i x_i \\ & x_i \geq 0 && \forall i \end{aligned}$$

Minimax Theorem

$$\Updownarrow$$

For any matrix A the following two LPs have the same optimal value:

$$\begin{aligned}
\max \quad & c \\
\text{subto} \quad & c \le \sum_i x_i a_{ij} && \forall j \\
& 1 = \sum_i x_i \\
& x_i \ge 0 && \forall i
\end{aligned}$$

$$\begin{aligned}
\min \quad & c \\
\text{subto} \quad & c \ge \sum_j y_j a_{ij} && \forall i \\
& 1 = \sum_j y_j \\
& y_j \ge 0 && \forall j
\end{aligned}$$

$$\Updownarrow$$

LP Duality