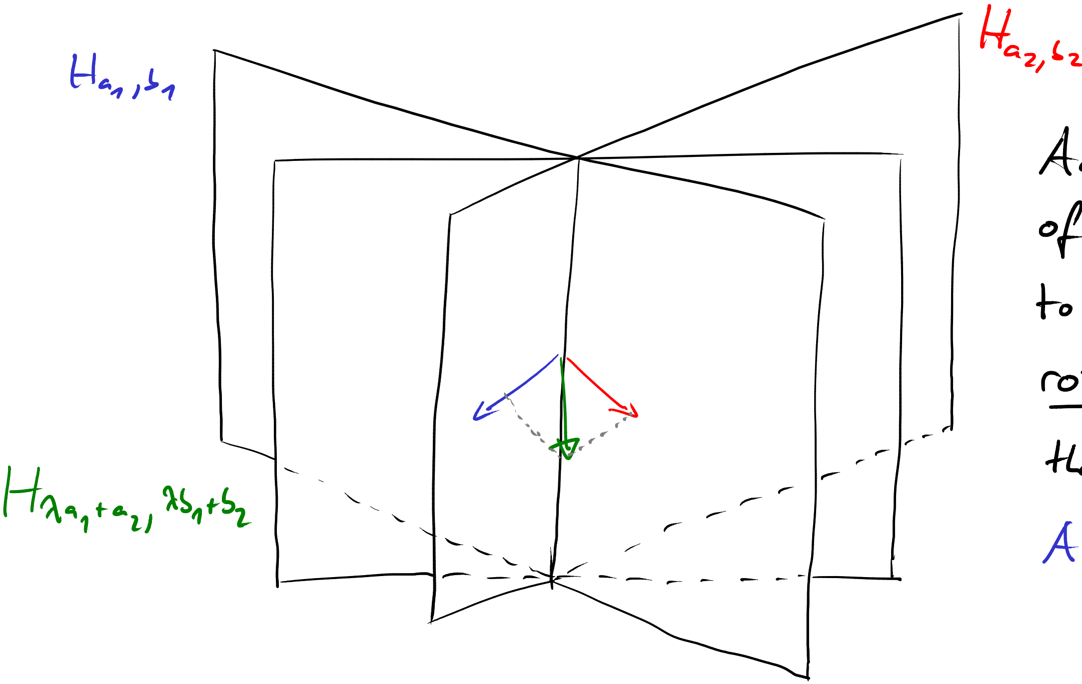
$$\langle x, a_1 \rangle = b_1$$
$$\langle x, a_2 \rangle = b_2$$

$\longrightarrow$

$$\langle x, a_1 \rangle = b_1$$
$$\langle x, \lambda a_1 + a_2 \rangle = \lambda b_1 + b_2$$



Adding a multiple of hyperplane A to hyperplane B rotates B around the intersection of A and B.

Lemma Let $a_1,\dots,a_k\in\mathbb{R}^d$, $b_1,\dots,b_k\in\mathbb{R}$, $\lambda_1,\dots,\lambda_k\in\mathbb{R}$. Then

$$\bigcap_i H_{a_i,b_i} \subset H_{\sum\lambda_i a_i,\, b_i}$$

Proof: $\langle x,a_i\rangle = b_i \ \forall i \Rightarrow \sum\lambda_i\langle x,a_i\rangle = \langle x,\sum\lambda_i a_i\rangle = \sum\lambda_i b_i$

Lemma $0\neq a_1,a_2\in\mathbb{R}^d$, $b_1,b_2\in\mathbb{R}$, $0\neq\lambda_1,\lambda_2\in\mathbb{R}$, $\sum\lambda_i a_i\neq 0$.

$$H_{a_1,b_1}\cap H_{a_2,b_2} = H_{a_1,b_1}\cap H_{\sum\lambda_i a_i,\,\sum\lambda_i b_i} = H_{a_2,b_2}\cap H_{\sum\lambda_i a_i,\,\sum\lambda_i b_i}$$

Proof: Switch roles!

$$a_2 = \frac{1}{\lambda_2}(\lambda_1 a_1 + \lambda_2 a_2) - \frac{\lambda_1}{\lambda_2}a_1$$

$$b_2 = \frac{1}{\lambda_2}(\lambda_1 b_1 + \lambda_2 b_2) - \frac{\lambda_1}{\lambda_2}b_2$$

$$H_{a_1,b_1} \cap H_{a_2,b_2} = H_{a_1,b_1} \cap H_{\sum \lambda_i a_i, \sum \lambda_i b_i} = H_{a_2,b_2} \cap H_{\sum \lambda_i a_i, \sum \lambda_i b_i}$$

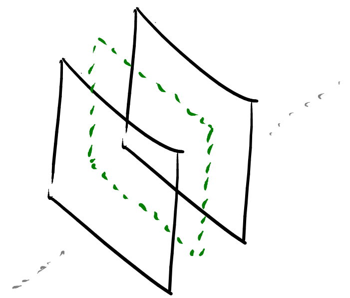
If a_1, a_2 are independent:

these are translates of the $(d-2)$-dimensional linear subspace orthogonal to a_1 and a_2.

$\sum \lambda_i a_i = 0 \Rightarrow \lambda_i = 0$

If $a_1 = \beta a_2$ but $b_1 \neq \beta b_2$:

these are all empty!



$\sum \lambda_i a_i = 0$

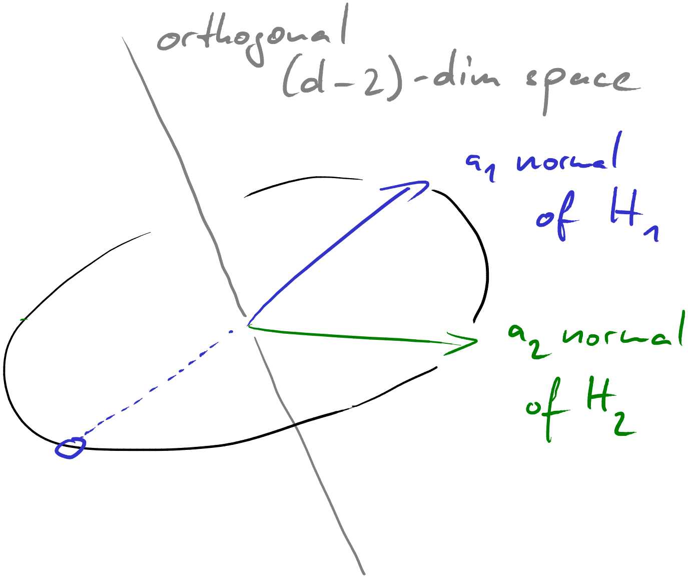
$\Rightarrow$ degen. hyperp. $= \emptyset$

$\sum \lambda_i a_i = 0$

$\Rightarrow$ degen. hyperp. $= \mathbb{R}^d$

If $a_1 = \beta a_2$ and $b_1 = \beta b_2$:

these are all the same $(d-1)$-dimensional hyperplane!



Q: Where can we turn the normal $\sum \lambda_i a_i$ to without changing the intersection?

$$H_1 \cap H_2 = H_1 \cap H_{\sum \lambda_i a_i, \sum \lambda_i b_i}$$

$$\iff \lambda_2 \neq 0.$$

<u>Observation</u> If H' is a third hyperplane with $a_1 \notin H'$, then there exist λ_1, λ_2 such that

$$H_1 \cap H_2 = H_1 \cap H_{\sum \lambda_i a_i, \sum \lambda_i b_i}$$

i.e. $\begin{array}{l} \langle x, a_1 \rangle = b_1 \\ \langle x, a_2 \rangle = b_2 \end{array} \iff \begin{array}{l} \langle x, a_1 \rangle = b_1 \\ \langle x, \sum \lambda_i a_i \rangle = \sum \lambda_i b_i \end{array}$

and $\sum \lambda_i a_i \in H.$

Gaussian Elimination

① pick a hyperplane H

② pick a coordinate such that the normal does not lie in the coordinate hyperp. $\{x \mid x_i = 0\}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

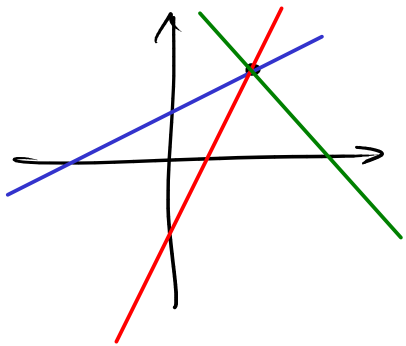
③ rotate all other hyperplanes around their intersection with H until their normals lie in the coordinate hyperplane $\{x \mid x_i = 0\}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2' \\ b_3' \end{pmatrix}$$

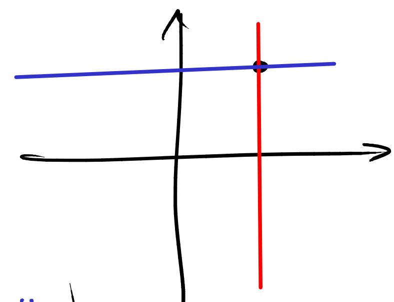
④ repeat until all normals point in coordinate directions

⑤ read off solution.

Example

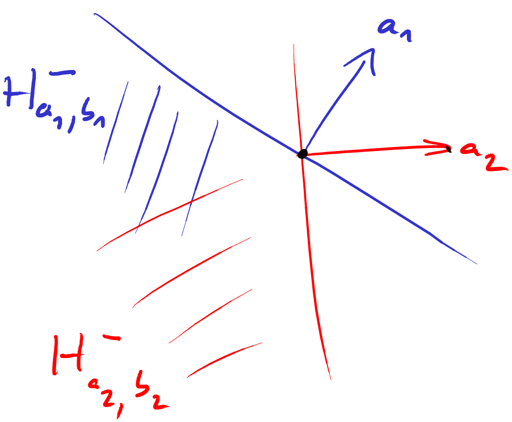


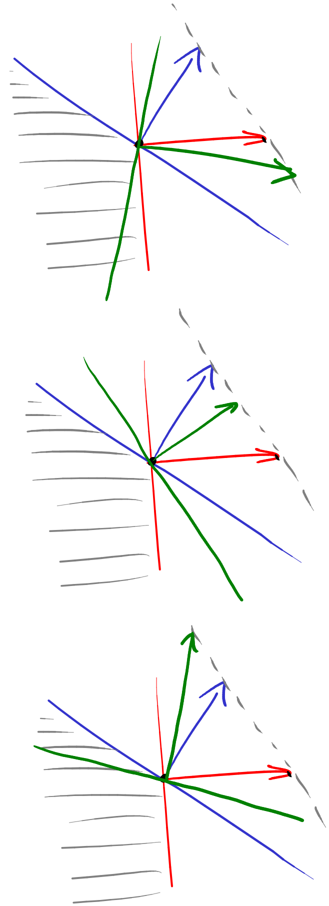
$$\left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right) \qquad \left(\begin{array}{cc|c} * & * & * \\ 0 & * & * \\ 0 & * & * \end{array}\right) \qquad \left(\begin{array}{cc|c} * & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{array}\right)$$



$$\left(\begin{array}{cc|c} * & 0 & * \\ 0 & * & * \\ 0 & 0 & 0 \end{array}\right)$$

Halfspaces





Observation: If

$\lambda_i \geq 0, \quad \sum \lambda_i = 1,$

then

$$H^-_{\sum \lambda_i a_i, \sum \lambda_i b_i} \supset \bigcap_i H^-_{a_i, b_i}$$

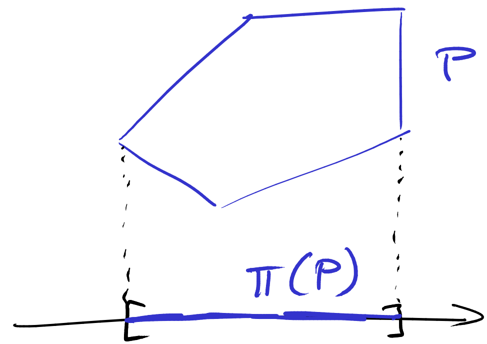
Proof: $Ax \leq b \Rightarrow \langle x, \lambda_i a_i \rangle \leq \lambda_i b_i \ \forall i \Rightarrow \langle x, \sum \lambda_i a_i \rangle \leq \sum \lambda_i b_i$

Q: When does $Ax \le b$ have a solution?

Idea: Induction on the dimension!

$$\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$$

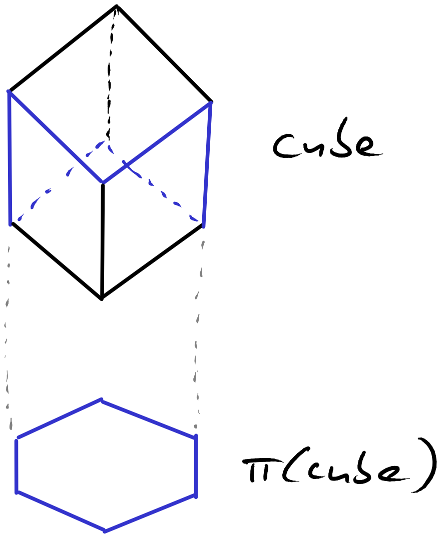
$$\begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix}$$



Observation P is non-empty $\iff \pi(P)$ is non-empty

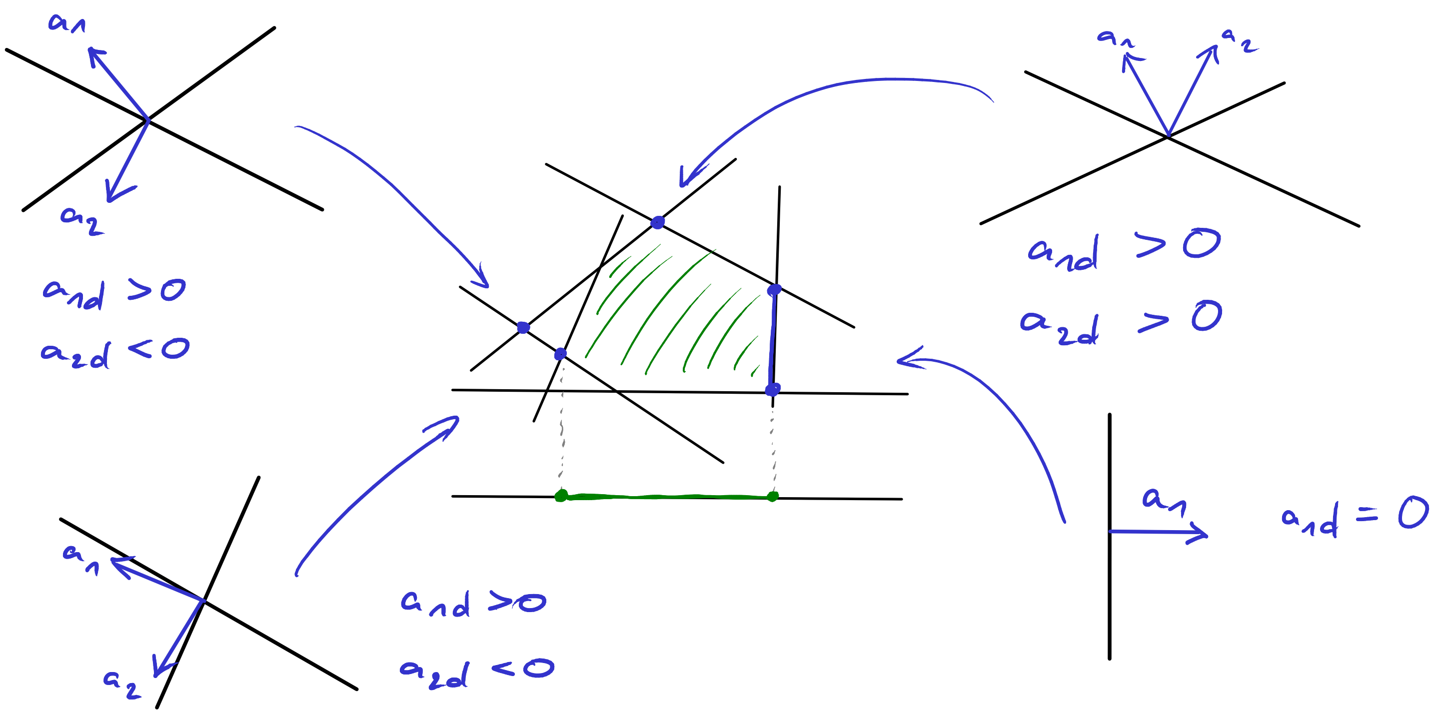
Base Case $d=1$: Easy!

How to transform $Ax \leq b$ into an inequality description of $\pi(P)$?



Facets of the projection are given by certain ridges of the original.

Which ridges?



Constraints on $\pi(P)$ are given by constraints on P of the form

① $\langle x, a_1 \rangle \leq b_1$ with $a_{1d} > 0$
$\langle x, a_2 \rangle \leq b_2$ with $a_{2d} < 0$

OR

② $\langle x, a_1 \rangle \leq b_1$ with $a_{1d} = 0$.

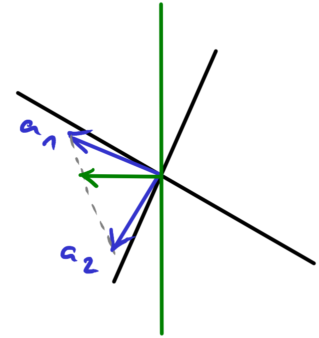
These need to be rewritten!

These are good!

How to turn this into an inequality defined on $\mathbb{R}^{d-1}$?

① $\langle x, a_1 \rangle \leq b_1$ with $a_{1d} > 0$

$\langle x, a_2 \rangle \leq b_2$ with $a_{2d} < 0$



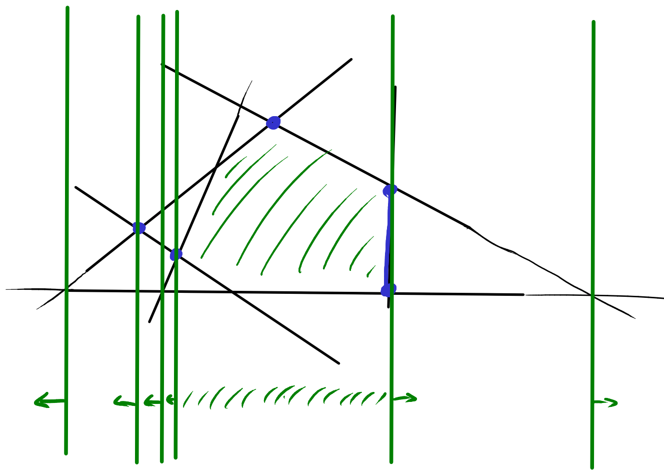
$\downarrow$

② $\langle x, \lambda_1 a_1 + \lambda_2 a_2 \rangle \leq \lambda_1 b_1 + \lambda_2 b_2$

with $\lambda_1, \lambda_2 > 0$

and $\left(\sum \lambda_i a_i\right)_d = 0$.

Note: If x satisfies ① then x satisfies ②!



Which pairs do we take?

All of them!

Fourier - Motzkin Elimination

<u>Thm:</u> If $P = \{x \mid Ax \leq b\}$ then the following is an inequality description of $\pi(P)$:

$\langle x, a \rangle \leq b$

for all rows $(a|b)$ of $(A|b)$ with $a_d = 0$

all "vertical" hyperplanes

$\langle x, -\frac{1}{a_{1d}} a_1 + \frac{1}{a_{2d}} a_2 \rangle \leq -\frac{1}{a_{1d}} b_1 + \frac{1}{a_{2d}} b_2$

for all pairs of rows $(a_1|b_1)$, $(a_2|b_2)$ of $(A|b)$ such that

$a_{1d} < 0$ and $a_{2d} > 0$.

for any potential "boundary ridge" given by hyperp. H_1, H_2 the vertical hyperplane obtained by turning H_1 towards H_2.

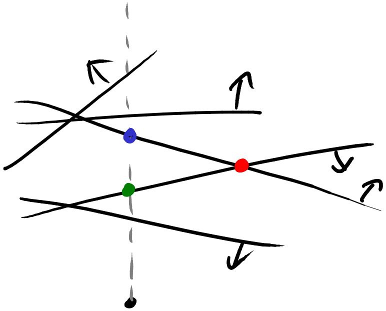
<u>Proof:</u> Let T be the polytope given by the inequality description. Want to show $T = \pi(P)$.

1) $T \supset \pi(P)$. ✓

- we used only some (pairs of) half-spaces
- taking "non-negative combinations" only makes the solution set larger!

2) $T \subset \pi(P)$?

Let $(x_1, \dots, x_{d-1}) \in T$. Does there exist a x_d such that $(x_1, \dots, x_d) \in P$?



$$\langle x, a \rangle \le b$$

$$\Leftrightarrow \begin{cases} x_d \le \frac{1}{a_d}\left(b - \sum_{i \ne d} x_i a_i\right) & \text{if } a_d > 0 \\ x_d \ge \frac{1}{a_d}\left(b - \sum_{i \ne d} x_i a_i\right) & \text{if } a_d < 0 \end{cases}$$

There exists an x_d such that $x \in P$.

$$\Leftrightarrow \frac{1}{a_{1d}}\left(b_1 - \sum_{i \ne d} x_i a_{1i}\right) \le \frac{1}{a_{2d}}\left(b_2 - \sum_{i \ne d} x_i a_{2i}\right)$$

for all $(a_1 | b_1), (a_2 | b_2)$ with $a_{1d} < 0$, $a_{2d} > 0$.

$$\Leftrightarrow \left\langle x, -\frac{1}{a_{1d}} a_1 + \frac{1}{a_{2d}} a_2 \right\rangle \le -\frac{1}{a_{1d}} b_1 + \frac{1}{a_{2d}} b_2$$

for all $(a_1 | b_1), (a_2 | b_2)$ with $a_{1d} < 0$, $a_{2d} > 0$.

$\square$

Corollary: Every V-polyhedron is an H-polyhedron.

Proof: $\operatorname{conv}(V) + \operatorname{cone}(Y)$

$$= \left\{ x \in \mathbb{R}^d \;\middle|\; \exists p \in \mathbb{R}^{|V|} \; \exists q \in \mathbb{R}^{|Y|} : \begin{array}{l} x = Vp + Yq \\ p \geq 0, \; q \geq 0, \; \sum p_i = 1 \end{array} \right\}$$

$$= \pi\left(\left\{ (x, p, q) \in \mathbb{R}^{d+|V|+|Y|} \;\middle|\; \begin{array}{l} x = Vp + Yq \\ p \geq 0, \; q \geq 0, \; \sum p_i = 1 \end{array} \right\} \right)$$

$$\pi : \mathbb{R}^{d+|V|+|Y|} \longrightarrow \mathbb{R}^d \quad (x, p, q) \longmapsto x.$$

▷ $\operatorname{cone}(V) + \operatorname{cone}(Y)$ is the projection of an H-polyhedron.

▷ By F-M: Projections of H-polyhedra are H-polyhedra.

□

Fourier–Motzkin Elimination allows us to

- ▷ convert V- to H-descriptions.
- ▷ check whether $Ax \le b$ has a solution
- ▷ find a solution if one exists
  - find $(x_1, \dots, x_{d-1}) \in \pi(P)$
  - find x_d
- ▷ enumerate all integer solutions

But the running time is terrible:

- ▷ each step essentially squares the number of inequalities.

Q: Given $Ax \leq b$.

If there <u>is</u> a solution, I can give you a valid x to prove this claim.

If there is <u>no</u> solution, is there some kind of <u>certificate</u> I can use to prove this claim?

Ansatz: How does Fourier-Motzkin show that there is no solution to $Ax \le b$.

Unsolvable $d=1$ ineq. system: $x_1 \le c_1 < c_2 \le x_1$

$$\Longleftrightarrow \quad a_1' x_1 \le b_1', \quad a_1' > 0, \quad \frac{b_1'}{a_1'} < \frac{b_2'}{a_2'}$$
$$a_2' x_1 \le b_2', \quad a_2' < 0,$$

$$\Rightarrow \quad 0 = \frac{a_1'}{a_1'} x_1 - \frac{a_2'}{a_2'} x_1 \le \frac{b_1'}{a_1'} - \frac{b_2'}{a_2'} < 0$$

By construction this inequality is a non-negative combination of inequalities in $Ax \le b$!

$\Rightarrow$ There exist $\lambda_1, \dots, \lambda_k$:

$$\lambda_i \ge 0, \quad \sum \lambda_i a_i = 0, \quad \sum \lambda_i b_i < 0$$

Farkas-Lemma

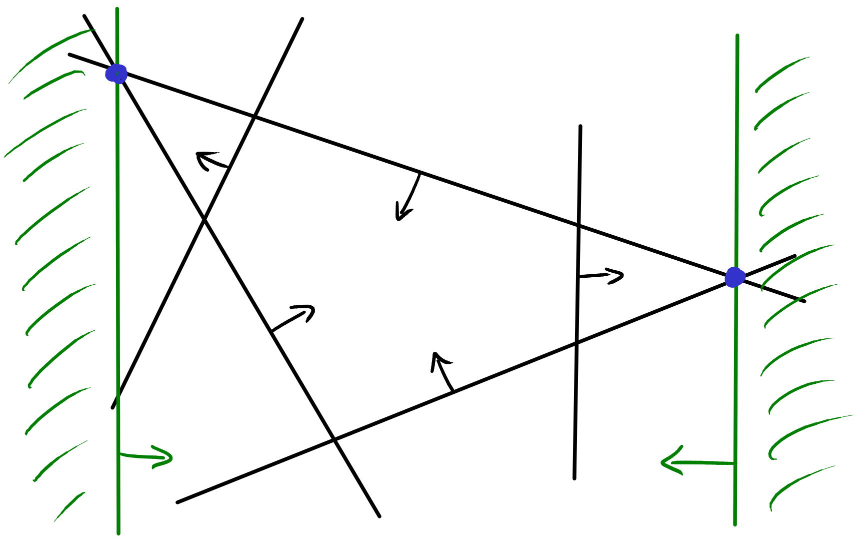
Either there exists x with $Ax \leq b$

or there exists λ with

$$\lambda \geq 0, \quad \lambda A = 0, \quad \lambda b < 0.$$

x: point in P

λ: certificate that there is no point in P.



Proof:

$\neg 1 \Rightarrow 2$: ✓

$\neg(1 \wedge 2)$:

$$Ax \leq b$$

$$\Rightarrow \lambda A x \leq \lambda b$$

$$\Rightarrow 0 < 0 \quad \lightning$$

$\square$

Farkas-Lemma is surprising!

- Usually $\neg(\exists x : \varphi(x)) \iff \forall x : \neg\varphi(x)$ is best possible.
- Here: $\neg(\exists x : \varphi(x)) \iff \exists y : \psi(x)$
- there exist certificates <u>both</u> for having a solution <u>and</u> for having no solution.

- both claims can be <u>verified</u> in polynomial time even though running F-M is exponential time.
- deciding solvability of $Ax \le b$ is in

$$NP \cap coNP$$

polynomial time verifiable problems

problems whose negation is polynomial time verifiable.

- **Thm:** Solving $Ax \le b$ is even in P. ← polynomial time solvable problems
- We are going to see the simplex algorithm for solving LPs, which is much faster than F-M.
- The simplex algorithm is not in P.
- LPs can be solved in polynomial time by, e.g., interior point methods.
- IPs, $\exists z \in \mathbb{Z}^d : Az \le b$, are NP-complete

as hard as any problem that is polynomial time verifyable.

This means that (unless $P = NP$)

$$\exists x : Ax \leq b$$

is much easier to solve than, e.g.,

$$\exists x : (x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_5 \lor \neg x_6) \land \ldots$$

because of the geometry of the problem!