Brouwer's Fixpoint Theorem

$X \subset \mathbb{R}^n$ convex, compact.

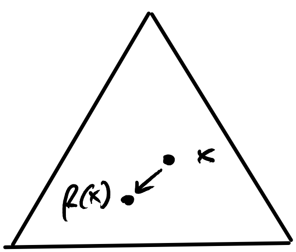
$f: X \to X$ continuous has a fixed point

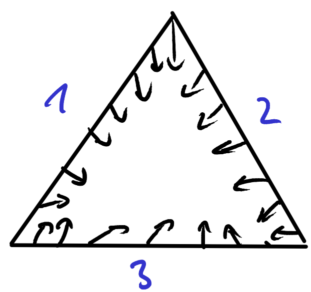
$\Leftrightarrow$ vector field $g: X \to \mathbb{R}^n$ with $x + g(x) \in X$ has a zero.

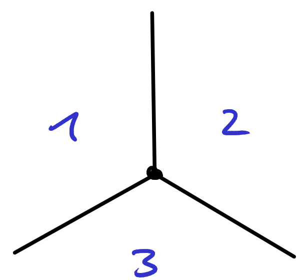
Sperner's Lemma

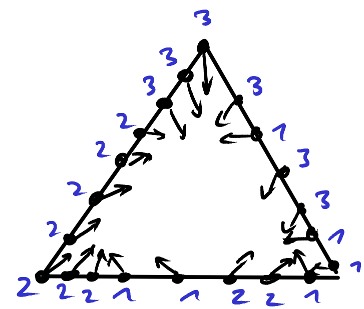
K a triangulation of Δ^n

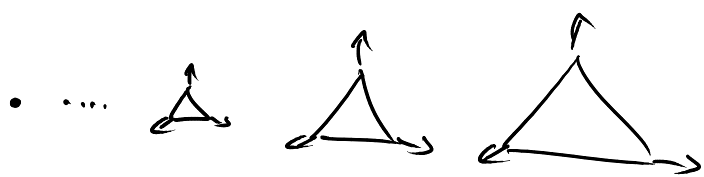
Every coloring $c: V(K) \to \{1, \dots, n+1\}$ with $c(v) \neq i$ for v in "side i" of Δ^n has an n-face labelled with *all* colors.











Fair Division

one cake $I = [1, 0]$, n people $i = 1, \dots, n$

preferences: μ_i a continuous probability measure on I

division: $I = A_1 \,\dot\cup\, A_2 \,\dot\cup \cdots \dot\cup\, A_n$, $\pi \in S_n$ a permutation

$$\boxed{\text{person } i \text{ gets } A_{\pi(i)}}$$

division is fair if $\mu_i(A_{\pi(i)}) \geq \frac{1}{n}$ $\forall i$

division is envy-free if $\mu_i(A_{\pi(i)}) \geq \mu_i(A_j)$ $\forall i, j$

Fair Cake Cutting

1) slowly move a knife along the cake
2) if person i yells STOP when the knife is at position $t \in [0,1]$, person i receives interval $[0,t]$.
3) divide the rest of the cake among the remaining $n-1$ people by the same procedure.

▷ it is in everyone's best interest to yell STOP as soon as $\mu_i([0,t]) \geq \frac{1}{n}$.

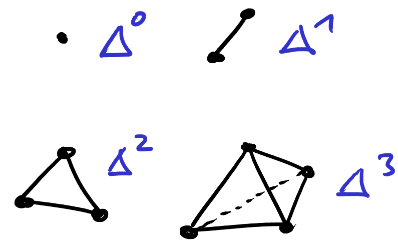
▷ the resulting division is fair!

Envy-free Cake Cutting

Theorem There exists an <u>envy-free division</u> $(A_1, \dots, A_n; \pi)$ such that the A_i are <u>intervals</u>.

- ▷ Proof via Sperner's Lemma.
- ▷ Yields a <u>constructive procedure</u> for approximating the envy-free division.
- ▷ Implementation by Francis Su: "Fair Division Calculator."

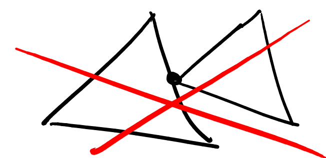
__n-simplex__ Δ^n: convex hull of $n+1$ affinely indep. points



__simplicial complex__ K: a set of simplices such that:

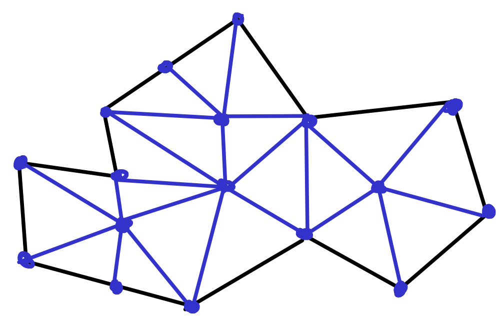
1) $\sigma \in K \wedge \sigma'$ a face of $\sigma \Rightarrow \sigma' \in K$.

2) $\sigma_1, \sigma_2 \in K \Rightarrow \sigma_1 \cap \sigma_2 \in K$ and $\sigma_1 \cap \sigma_2$ is a face of both σ_i

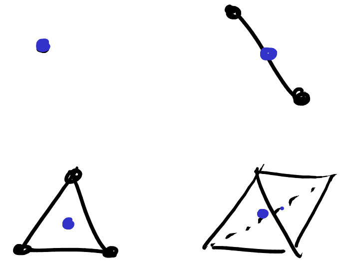


K is a __triangulation__ of $X \subset \mathbb{R}^n$:

K is a simplicial complex and $\bigcup_{\sigma \in K} \sigma = X$



the <u>barycenter</u> of a simplex σ is

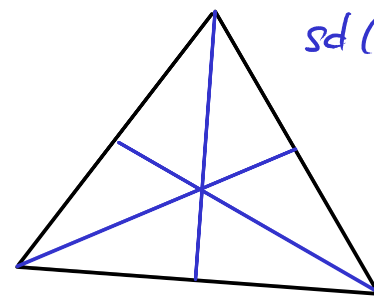


$$b(\sigma) = \frac{1}{n}\sum_{i=1}^{n} v_i$$

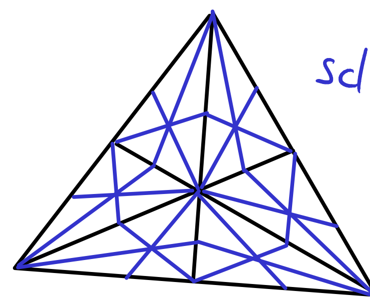
where $v_1, \dots, v_n$ are the vertices of σ.

<u>barycentric subdivision</u> of a complex K is

$$sd(K) = \{ \mathrm{conv}(b(\sigma_1), \dots, b(\sigma_k)) : \sigma_1, \dots, \sigma_k \in K \setminus \{\emptyset\}, \; \sigma_1 \subset \dots \subset \sigma_k \}$$



$sd(\Delta^2)$



$sd^2(\Delta^2)$

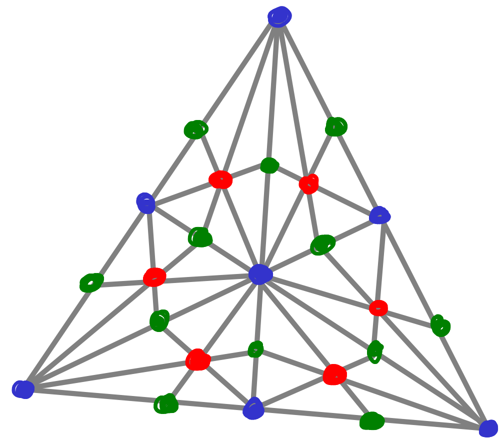
$$sd^m(K) = \underbrace{sd(sd(\dots sd(K)))}_{m \text{ times}}$$

Theorem There exists an envy-free division $(A_1, \ldots, A_n; \pi)$ such that the A_i are intervals.

(1) label vertices with persons

$$p : V(sd^m(\Delta^{n-1})) \to \{1, \ldots, n\}$$

s.t. every $(n-1)$-simplex $\sigma \in sd^m(\Delta^{n-1})$ gets all labels $1, \ldots, n$.

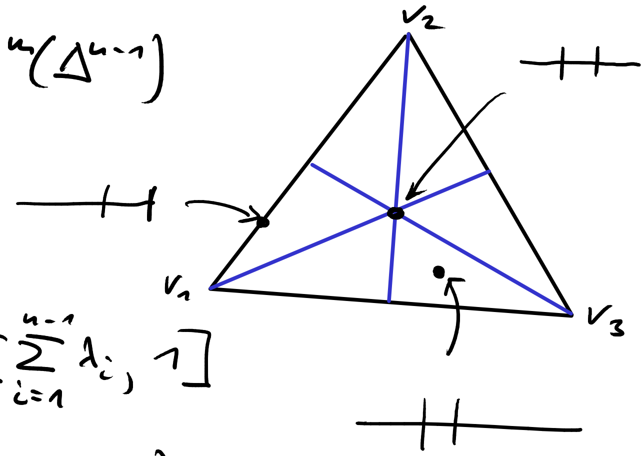


② Identify vertices v of $sd^n(\Delta^{n-1})$

$$v = \sum_{i=1}^{n} \lambda_i v_i$$

with divisions

$$\underbrace{[0, \lambda_1]}_{I_1(v)} \dot\cup \underbrace{[\lambda_1, \lambda_1 + \lambda_2]}_{I_2(v)} \dot\cup \dots \dot\cup \underbrace{[\sum_{i=1}^{n-1} \lambda_i, 1]}_{I_n(v)}$$



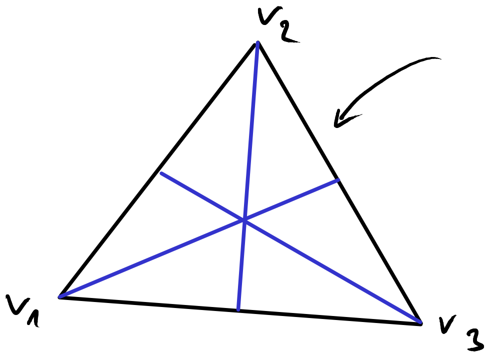
③ At vertex v of $sd^n(\Delta^{n-1})$, ask person i which piece they want!

$$\alpha(v) = \operatorname{argmax}_j \mu_{p(v)}(I_j(v))$$

$$\mu(v) = \max\{\mu_{p(v)}(I_j(v)) : j = 1, \dots, n\}$$

$$\alpha(v) = \min\{j : \mu(v) = \mu_{p(v)}(I_j(v)), j = 1, \dots, n\}$$

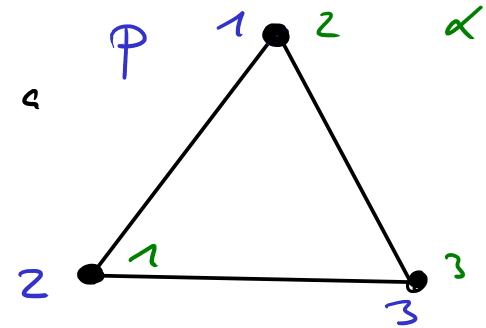
4. α is a Sperner labeling!



on this side, the first interval is empty!

nobody is going to pick the first interval!

5. By Sperner's Lemma, there is a fully α-labelled simplex.



Each person likes a different piece best!

6. Limit argument!

$\square$

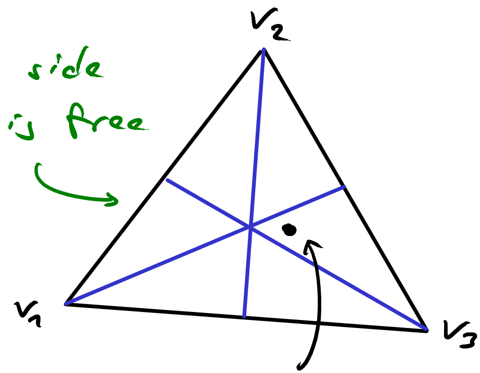
Rent Partitioning

1 apartment, 3 rooms, 3 tenants

\$2000 total rent

At each vertex $v \in sd^n(\Delta^2)$, ask tenant $p(v)$ which room they would choose if the rent partition is $\lambda(v)$.

on this side room 3 is free



$\$600\, v_1 + \$650\, v_2 + \$750\, v_3$

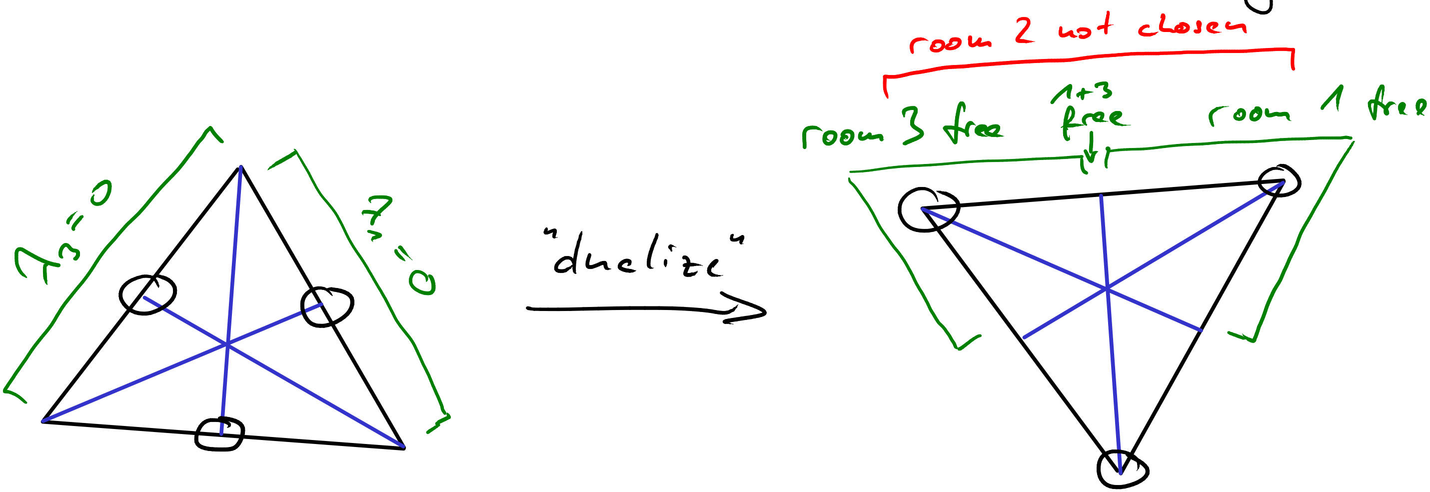
cost of room 1

▷ α is not a Sperner labeling!

▷ But: on side $\lambda_3 = 0$, everybody will prefer room 3 over a room that costs something!

At each vertex $v \in sd^n(\Delta^2)$, ask tenant $p(v)$ which room they would choose if the rent partition is $\lambda(v)$.

- α is <u>not</u> a Sperner labeling!
- But: on side $\lambda_i = 0$, <u>everybody</u> will prefer room i over a room that costs something!



- "Fair Division Calculator"

Detour: Median Hyperplanes and Robust Statistics

Robust Statistics: Which estimators are resistent to outliers?

An *estimator of location* (in dimension 1) is a map M that takes a finite sequence of real numbers $v_1, \ldots, v_n$ to a single real number (the estimate).

Examples: Mean, Barycenter, Median, Quantiles

The <u>breakpoint</u> of M is the infimum of all numbers $\lambda \in (0, \frac{1}{2}]$ such that there exists a sequence $(v_1^i, \dots, v_n^i)_i$ of n-element sets such that

- ▷ at most $\lambda \cdot n$ of these sequences are non-constant
- ▷ $\left(M(v_1^i, \dots, v_n^i)\right)_i$ is unbounded.

<u>Informally</u>: How many outliers do you need to make the estimate arbitrarily bad?

- ▷ <u>Mean</u> and <u>barycenter</u> have <u>breakpoint 0</u>!
- ▷ <u>Median</u> has <u>breakpoint</u> $\frac{1}{2}$!

▷ The median is a robust estimator of location.

→ "Outlier resistant"

▷ The median is the "best fit" to a set of samples when minimizing the "sum of errors." (As opposed to sum of squared errors!)

→ Exercise!

▷ The median can be computed by a linear program.

→ Exercise!

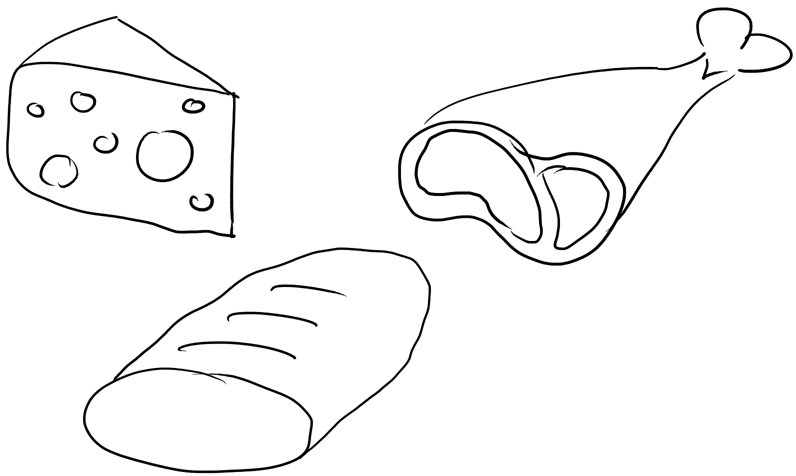
Ham Sandwich Splitting

$\mu_1, \dots, \mu_n$ continuous probability measures on $\mathbb{R}^n$.

$\lambda_1, \dots, \lambda_n \in (0,1)$.

Does there exist a hyperplane H with

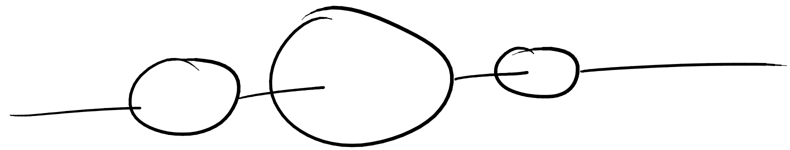
$$\mu_i(H^+) = \lambda_i \quad \text{for all } i \, ?$$



$\mu_1, \dots, \mu_n$ continuous probability measures on $\mathbb{R}^n$. $\lambda_1, \dots, \lambda_n \in (0,1)$.

Does there exist a hyperplane H with $\mu_i(H^+) = \lambda_i$ for all i?

In general: No!



Ham Sandwich Theorem

If $\lambda_i = \frac{1}{2} \; \forall i$, then yes!

App. of Borsuk-Ulam Thm.
$\to$ Seminar

Theorem

If μ_i "can be seperated", then yes!

App of Brouwer's Fixed Point Thm
$\to$ Today!

Intermediate Value Theorem

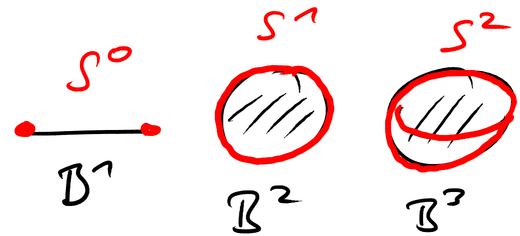
$f : [-1, 1] \to \mathbb{R}$ continuous.

If $f(-1) \le 0 \le f(1)$, then $f(x) = 0$ for some $x \in [-1, 1]$.

▷ Generalization to higher dimensions?

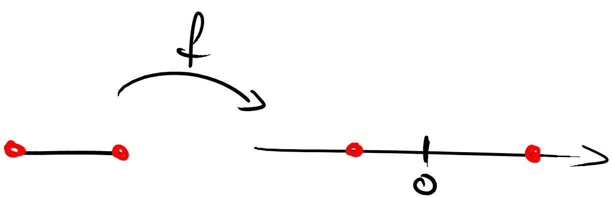
<u>sphere</u> $S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$

<u>ball</u> $B^n = \{ x \in \mathbb{R}^n \mid \|x\| \le 1 \}$



<u>Intermediate Value Thm</u> $f : B^1 \to \mathbb{R}^1$

"$f(S^0)$ around 0"



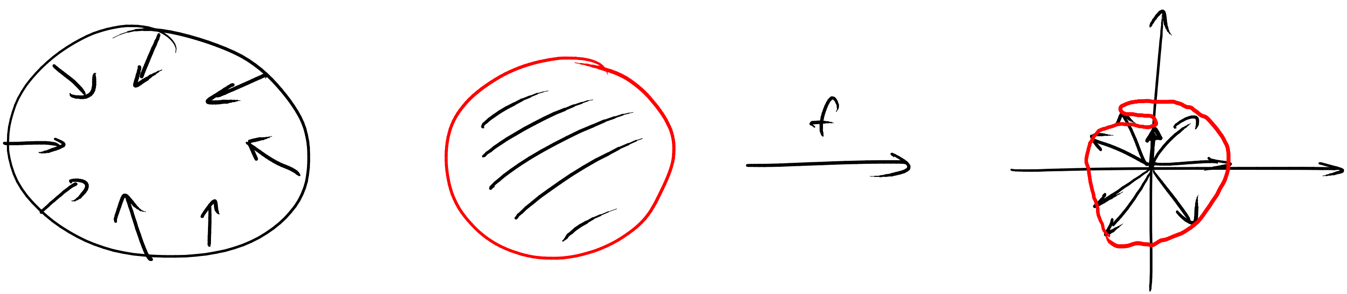
If $\boxed{f(-1) \le 0 \le f(1)}$

then $0 \in f(B^1)$

$f: B^n \to \mathbb{R}^n$ continuous

If $f(S^{n-1})$ is "around" 0,

then $f(B^n)$ contains 0.



Brouwer If $\overbrace{x + g(x) \in B^n \text{ for } x \in S^{n-1}}^{g(S^{n-1}) \text{ "around" } 0}$,

then $0 \in g(B^n)$.

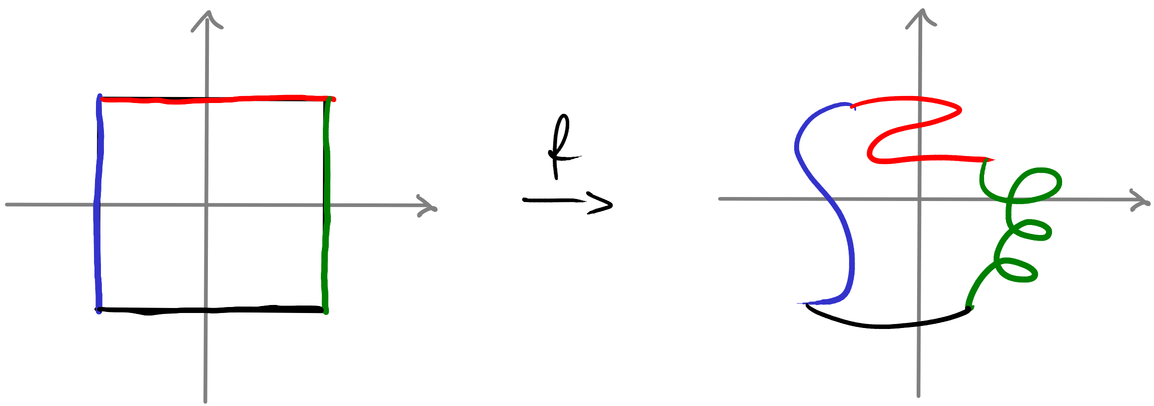
Poincaré - Miranda Theorem

$f : [-1,1]^n \to \mathbb{R}^n$ continuous such that

for every face σ of $[-1,1]^n$

$$\sigma \subset H^+ \implies f(\sigma) \subset H^+$$

for all coordinate half-spaces H^+. Then $0 \in \operatorname{Im} f$.



Uneven Ham Sandwich Splitting

Sets $S_1, \dots, S_n \in \mathbb{R}^n$ can be <u>seperated</u>, if, for every sign vector $\sigma \in \{-1,1\}^n$ there exists a hyperplane $H_{a,b}$ such that

$$\operatorname{int}\left(H_{a,b}^{\sigma(i)}\right) \supset S_i \qquad \forall i.$$

<u>Theorem</u> Let $\mu_1, \dots, \mu_n$ cont. prob. measures on $\mathbb{R}^n$ and $\alpha_1, \dots, \alpha_n \in [0,1]$. Let $S_1, \dots, S_n$ be bounded sets with the property that

$$\forall i \, \forall a \neq 0 \, \exists b_a : \mu_i(H_{a,b_a}^+) = \alpha_i \text{ and } H_{a,b_a}^+ \cap S_i \neq \emptyset.$$

If $S_1, \dots, S_n$ can be seperated, then there exists a hyperplane H with

$$\mu_i(H^+) = \alpha_i \quad \forall i.$$

Apply Poincaré-Miranda

$$f : (a, b) \longmapsto \left(\mu_i(H^+_{a,b}) - \alpha_i\right)_i$$

separation property $\rightarrow$ corners of the cube