Integral Flows

in $\mathbb{Z}$

(directed) multigraph

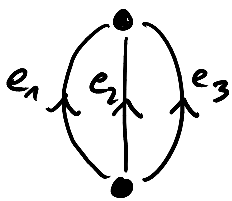
$\varphi(k)$ = # nowhere zero flows on G
w/ weights in $\{-k+1, \ldots, 0, \ldots, k-1\}$ ← flow in both directions

= # integer points in $k \cdot (-1,1)^E$
- in flow-space
- off coordinate hyperplanes

← linear subspaces

$$= \# \mathbb{Z}^E \cap k \cdot \left(P \setminus \bigcup H_i\right)$$

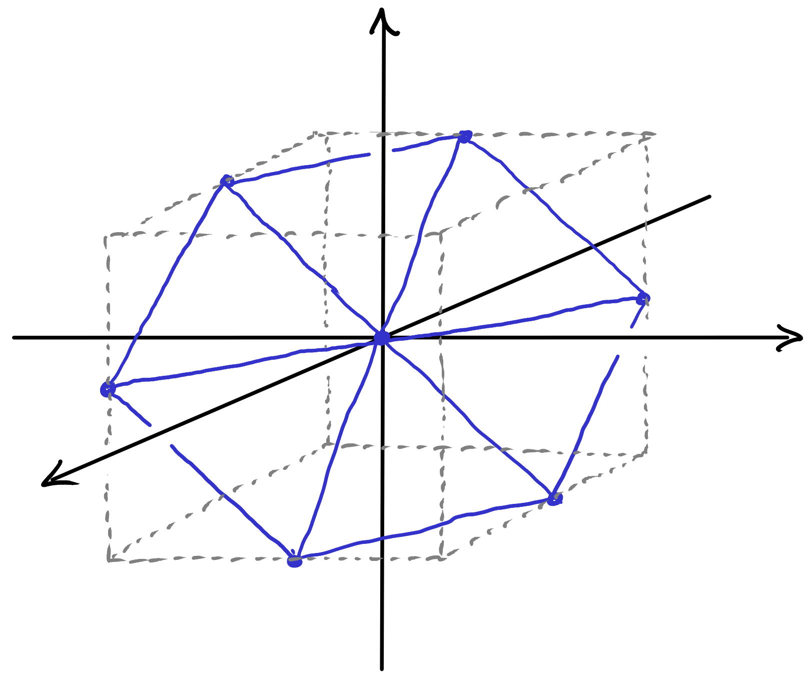
$$=: L_{P \setminus \bigcup H_i}(k)$$ ← Ehrhart function

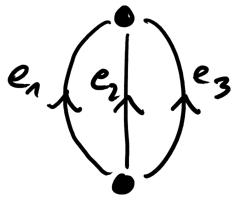


$$x \in (-1,1)^E$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1, x_2, x_3 \neq 0$$



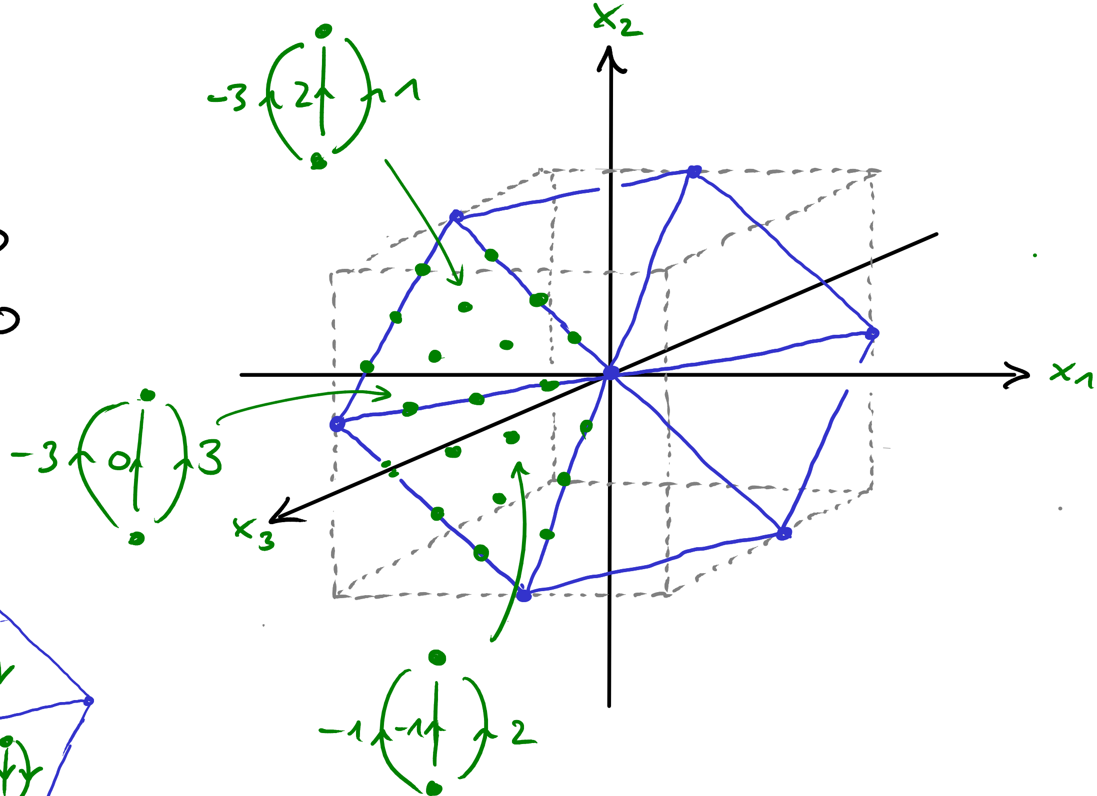


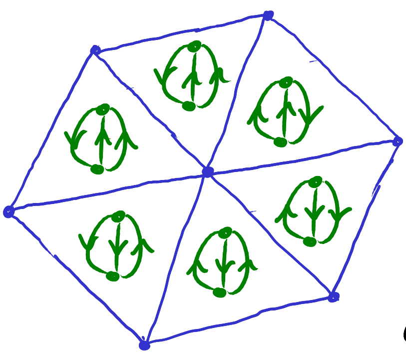
$$x \in (-1,1)^E$$

$$x_1 + x_2 + x_3 = 0$$

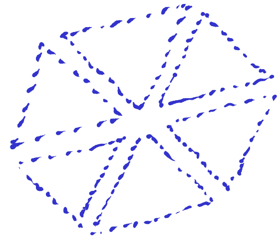
$$x_1, x_2, x_3 \neq 0$$

$k = 4$





components $\longleftrightarrow$ totally cyclic orientations

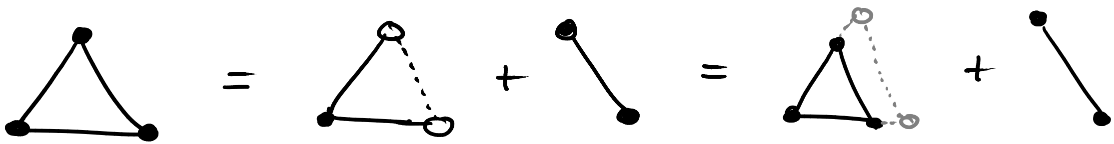
$P \setminus \bigcup_i H_i =$ 

$= \bigcup$ open lattice politopes
w/ unimodular triangulation

unimodular simplex $\cong \operatorname{conv}(e_1, \dots, e_d)$
up to $GL(\mathbb{Z})$ transformation

What is $L_\Delta(k)$ for Δ = unimodular simplex?

$$\Delta_i^d = \{ x \in \mathbb{R}^{d+1} \mid \sum_i x_i = 1,\ x_1, \dots, x_i > 0,\ x_{i+1}, \dots, x_{d+1} \geq 0 \}$$



$$k\Delta_0^d = k\Delta_1^d + k\Delta_0^{d-1} = (k-1)\Delta_0^d + k\Delta_0^{d-1}$$

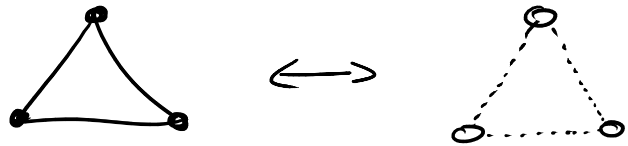
$$L_{\Delta_0^d}(k) = \binom{k+d}{d} \qquad L_{\Delta_i^d}(k) = \binom{k+d-i}{d}$$

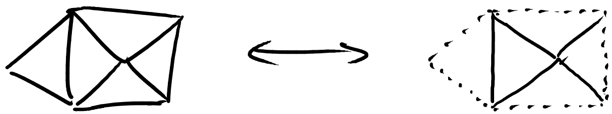
polynomials in k!

$$\Rightarrow \quad \varphi(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i} \quad \text{where } 0 \leq f_i \in \mathbb{Z}$$

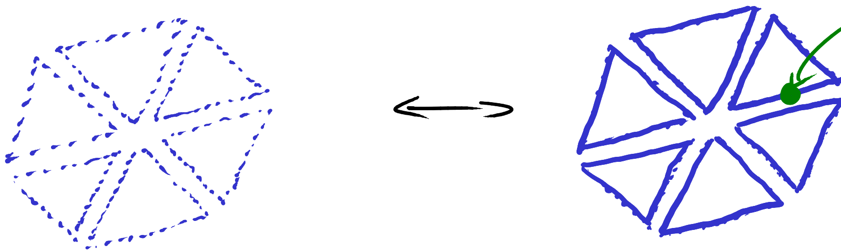
and $d = |E| - |V| + \#\text{components of } G$

$$L_{\Delta^d_i}(-k) = (-1)^d L_{\Delta^d_{d+1-i}}(k)$$





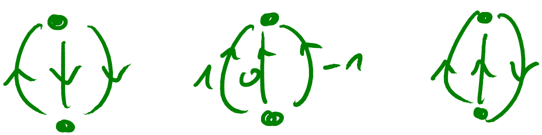
$\Rightarrow$



flow x counted once for each component x is contained in

$\Rightarrow \; |\varphi(-k)| = \#(x, o)$: x a $\{-k, \dots, k\}$-flow

o a compatible totally cyclic orientation



Integral Flows: Summary

$\varphi(k) = \#$ nowhere zero $\{-k+1, \dots, +k-1\}$-flows on G

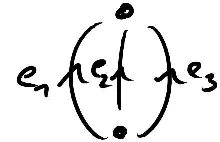
▷ is a polynomial (Kochol '02)

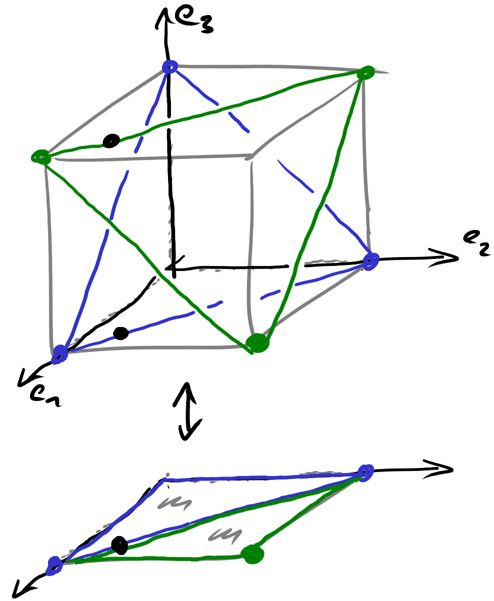
▷ of the form

$$\varphi(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i} \quad w/ \quad 0 \leq f_i \in \mathbb{Z}$$

▷ $|\varphi(-k)| = \#(x, o)$: x a $\{-k, \dots, +k\}$-flow on G (Beck, Zaslavsky)
o a compatible orientation.

Modular Flows

$G =$ 



Theorem (Sanyal, B.)

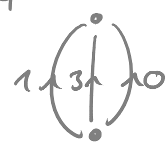
$\bar{\varphi}(k) = \#$ nowhere zero $\mathbb{Z}_k$-flows on G

$|\bar{\varphi}(-k)| = \#(x, o)$:

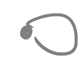
x a $\mathbb{Z}_k$-flow on G

o a totally cyclic orientation of $G/\mathrm{supp}\, x$

$k = 4$

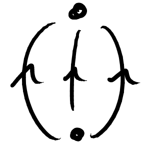
$x =$ 

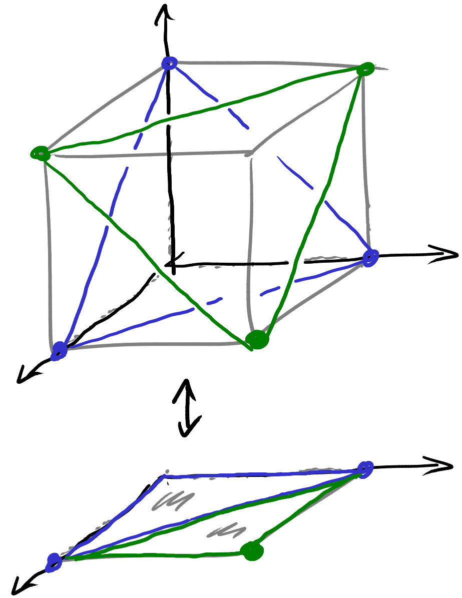
$G/\mathrm{supp}\, x$



two tot. cyc. orientations:

Theorem (Dall, B.)





▷ $\overline{\varphi}(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i}$

with $0 \le f_i \in \mathbb{Z}$

▷ $(k+1)^d - \overline{\varphi}(k) = \sum_{i=0}^{d} h_i \binom{k+d-i-1}{d-1}$

where $h_0 = 1$, $h_i \in \mathbb{Z}$ and

i) $h_0 \le h_1 \le \dots \le h_{\lfloor \frac{d}{2} \rfloor}$

ii) $h_i \le h_{d-i}$ for $i \le \frac{d}{2}$.

iii) $(h_0, h_1 - h_0, \dots, h_{\lceil \frac{d}{2} \rceil} - h_{\lceil \frac{d}{2} \rceil - 1})$ is an M-vector.

"g-constraints"

K = boundary complex of a simplicial polytope

Bruggesser & Mani $\quad$ K is shellable

Dehn & Sommerville $\quad$ $h_k = h_{d-k}$

g-Theorem $\quad$ $h_0, \dots, h_d$ is the h-vector of the boundary complex of a simplicial d-polytope if and only if

$$(h_0, h_1 - h_0, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1}) \text{ is an } M\text{-vector.}$$

M-vector $\quad$ For $0 < h, i \in \mathbb{Z}$ there exists a unique sequence of integers $a_i > a_{i-1} > \dots > a_j \geq j \geq 1$ such that $h = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \dots + \binom{a_j}{j}$

$$h^{\langle i \rangle} := \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \dots + \binom{a_j + 1}{j+1}$$

$(h_0, \dots, h_d)$ is an M-vector if $h_0 = 1$ and

$$h_{i+1} \leq h_i^{\langle i \rangle} \quad \text{for all } 1 \leq i \leq d-1.$$

Convex Ear Decompositions

Let K be a $(d-1)$-dimensional simplicial complex.

A c.e.d is a sequence $K_1, \dots, K_m$ of pure $(d-1)$-dim subcomplexes such that

i) K_1 is the boundary complex of a simp. polytope.
$\forall j \geq 2$: K_j is a top. ball that is a subcomplex of the boundary complex of some simp. polytope.

ii) $\partial K_j = K_j \cap \bigcup_{i<j} K_i$

iii) $K = \bigcup_i K_i$

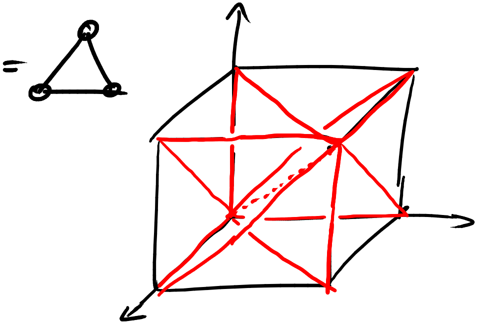
Theorem (Chari, Swartz) If K has a convex ear decomposition then its h-vector satisfies the g-constraints.

i) $h_0 \leq h_1 \leq \dots \leq h_{\lfloor \frac{d}{2} \rfloor}$

ii) $h_i \leq h_{d-i}$ for $i \leq \frac{d}{2}$.

iii) $(h_0, h_1 - h_0, \dots, h_{\lceil \frac{d}{2} \rceil} - h_{\lceil \frac{d}{2} \rceil - 1})$ is an M-vector.

Colorings of Hypergraphs

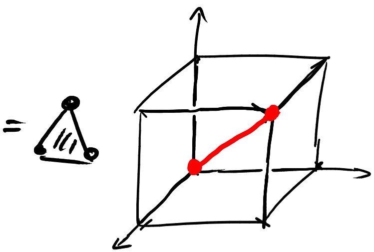
$G =$ 

$c: E \to [k]$ proper iff

$\forall e \in E \; \exists v,w \in e : c(v) \neq c(w)$

$\chi_G(k) = \#$ proper k-colorings of G

$$\chi_G(k+1) = L_{[0,1]^{|V|} \setminus \bigcup_{e\in E} \{c \mid c_v = c_w \; \forall v,w \in e\}}(k)$$

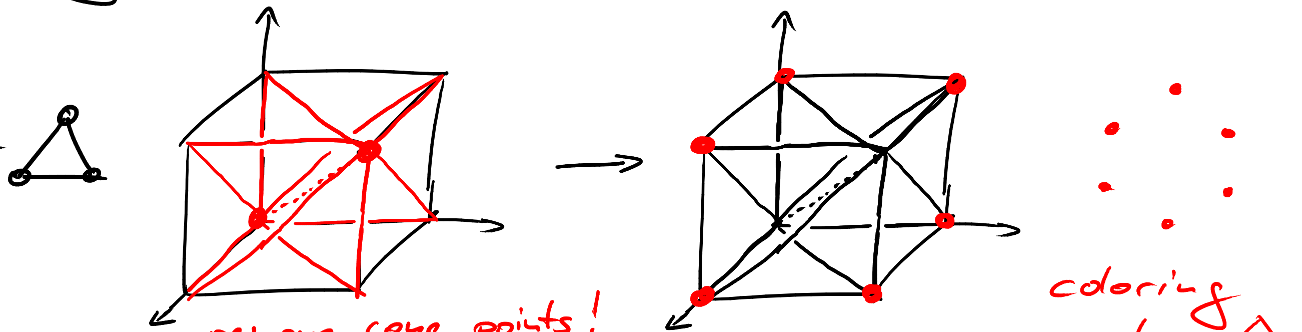
$G =$ 

$$\chi_G(k+1) = \sum_{i=0}^{|V|} f_i \binom{k-1}{i} \quad \text{w/} \quad 0 \le f_i \in \mathbb{Z} \quad \& \text{ more}$$

	graphs	hypergraphs
δ-constraints?	yes (Hersh, Swartz)	no ← (Dall, Kubitzke, B)
reciprocity?	yes (Stanley; Beck, Zaslavsky)	?

Coloring Complex

$G =$

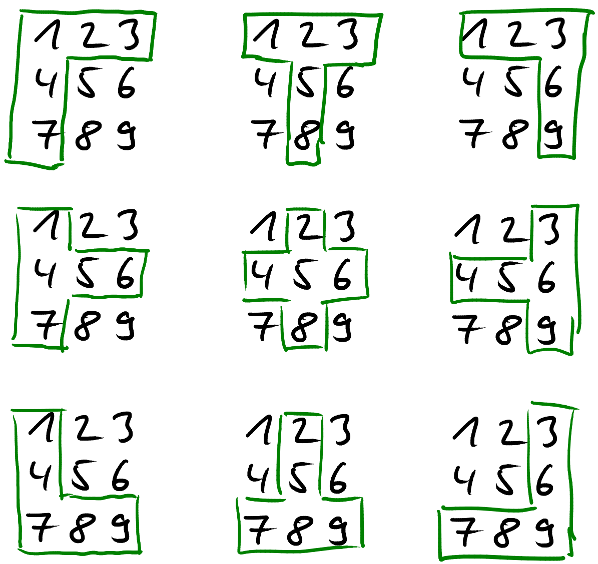


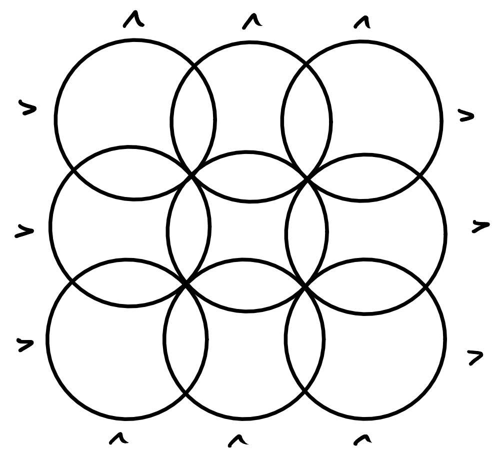
properties of Δ_G	graphs	(uniform) hypergraphs
Cohen-MacCaulay	yes (Jonsson)	no
shellable	yes (Hultman)	no
partitionable	yes	no
$h_i \geq 0$	yes	no
wedge of spheres	yes (Steingrímsson)	conj: no

(Dall, Kubitzka, B.)

__Conjecture__ The coloring complex of the following hypergraph does not have the homotopy type of a wedge of spheres.

$V = [9]$, $E =$





"edge spheres"

To which counting functions can Ehrhart methods be applied?

(Work In Progress)

Thm (B) For counting functions $f(k)$ the following are <u>equivalent</u>

1) $f(k)$ is a sum of counting functions of the form
$$\#\{x:[n]\to[k] \mid \varphi(x)\}$$
where φ is an arbitrary logical formula that only depends atomic formulas of the form "$x_i \leq x_j$".

2) $f(k)$ is the Ehrhart function of a disjoint union of relatively open lattice polytopes.

3) $f(k) = \sum_{i=0}^{d} f_i \binom{k-i}{i}$ for some $0 \leq f_i \in \mathbb{Z}$.

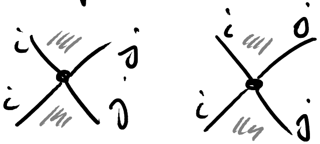
⌜In which case $f_i = \#x$ with $\varphi(x)$ and $|\mathrm{Im}\, x| = i$.⌟

Examples

$x : [n] \to [k] \iff$ ordered partitions of $[n]$ into k sets.

1) # order preserving maps $\Pi \to [k]$ (order polynomial)

2) # semistandard tableaux of fixed shape with max $\leq k$.

3) # $g : E \to [k]$ s.t. 

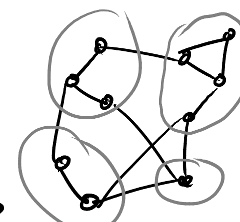
for a fixed medial graph of a planar map

(Penrose polynomial)

4) # ord partitions of V into k independent sets.

(chromatic polynomial)

5) # ord. partitions of V into k connected sets



6) # ord. partitions of V that induce an H-Minor of G