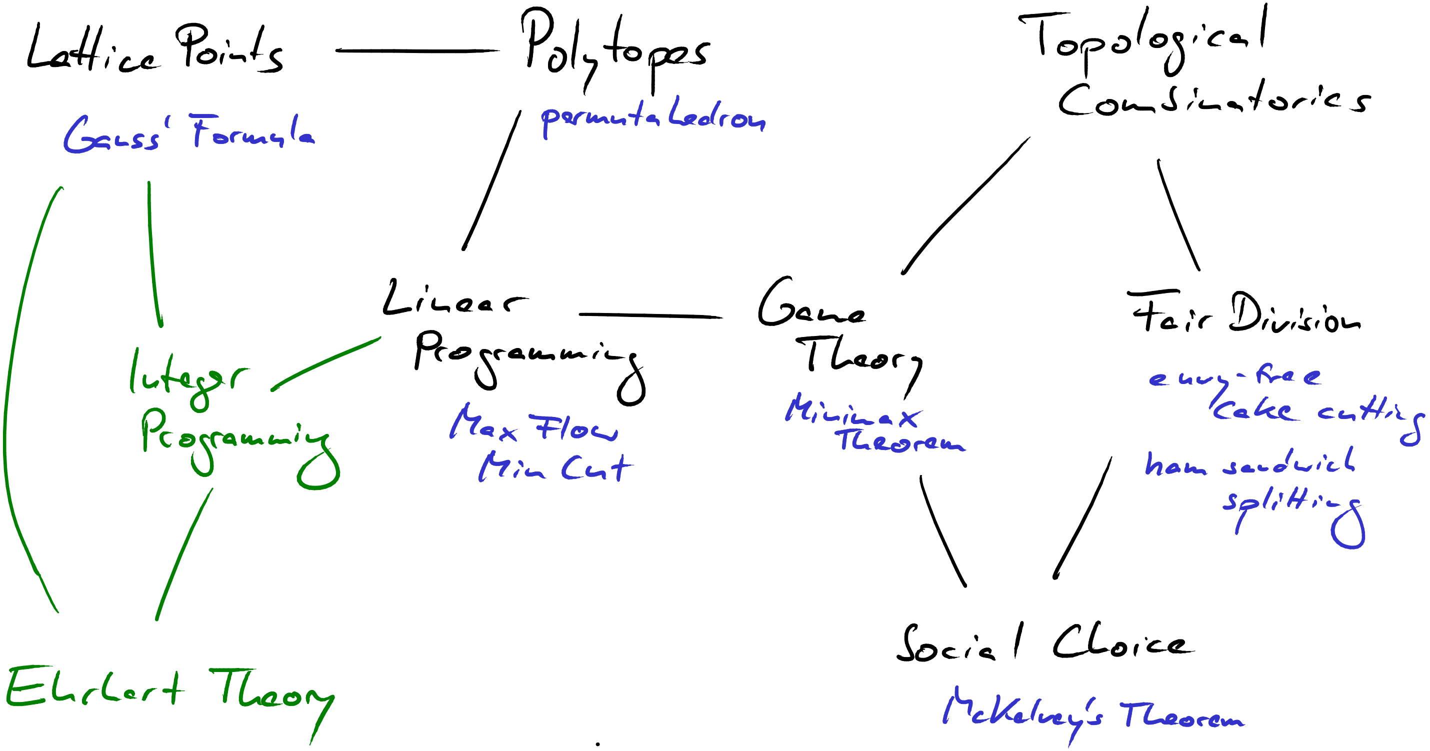


Lattice Points
Gauss' Formula
Polytopes
permutahedron
Topological Combinatorics
Linear Programming
Max Flow Min Cut
Game Theory
Minimax Theorem
Fair Division
envy-free cake cutting
ham sandwich splitting
Integer Programming
Ehrhart Theory
Social Choice
McKelvey's Theorem

Integer Programming

- $$\max_{x \in \mathbb{Z}^d} \; cx \qquad Ax \leq b$$

- Does there exist an $x \in \mathbb{Z}^d$ with $Ax \leq b$?

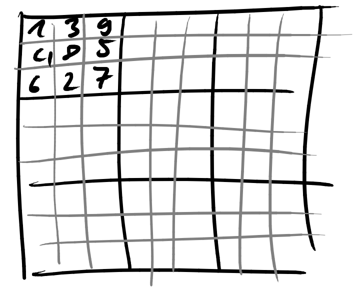
- These problems are hard (NP-complete) in contrast to linear programming.

- Nonetheless there exist algorithms that work well in practice!

- Combinatorial Optimization:
  - Model combinatorial problem as IP.
  - Apply IP methods to solve problem.

- (Toy)-Example: Sudoku.

Sudoku



boxes: $B = \overset{\text{rows}}{\{1,\dots,9\}} \times \overset{\text{cols}}{\{1,\dots,9\}}$

solution $f : B \to \{1,\dots,9\}$

$f|_R : R \to \{1,\dots,9\}$ is a bijection $\forall$ rows R

$f|_C : C \to \{1,\dots,9\}$ is a bijection $\forall$ cols C

$f|_S : S \to \{1,\dots,9\}$ is a bijection $\forall$ squares S

▷ How to model this as an IP ?

Sudoku

$x \in \mathbb{Z}^{9 \times 9 \times 9}$ ← in 729 dimensions!

← integer programming

$0 \leq x \leq 1$

$$\left.\begin{array}{ll} \sum x_{ijk} = 1 & \forall j \\ \sum x_{ijk} = 1 & \forall k \end{array}\right\} \forall i$$

$$\left.\begin{array}{ll} \sum x_{ijk} = 1 & \forall i \\ \sum x_{ijk} = 1 & \forall k \end{array}\right\} \forall j$$

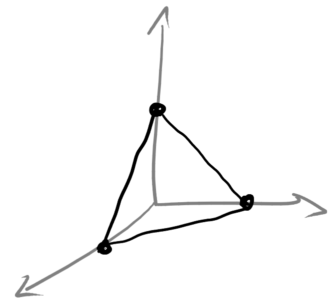
$$\left.\begin{array}{ll} \sum_{k} x_{ijk} = 1 & (i,j) \in S \\ \sum_{(i,j) \in S} x_{ijk} = 1 & \forall k \end{array}\right\} \forall \text{ squares } S$$

- ▷ The lattice points in this polytope are in bijection with solutions to Sudoku.
- ▷ Solving combinatorial puzzles often becomes easier by translating them into a geometric setting.
- ▷ This is why high-dimensional geometry is important!

Lattice Points in Simplices

$$\Delta^d = \{x\in\mathbb{R}^{d+1} \mid \sum x_i = 1,\ x_i \geq 0\}$$

$$= \mathrm{conv}(e_1,\dots,e_{d+1}) \subset \mathbb{R}^{d+1}$$

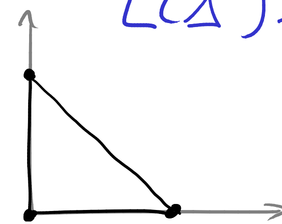


lattice transform

$$\cong \{x\in\mathbb{R}^d \mid \sum x_i \leq 1,\ x_i \geq 0\}$$

$$= \mathrm{conv}(0, e_1,\dots,e_d) \subset \mathbb{R}^d$$

$L(\Delta^d) = d+1$



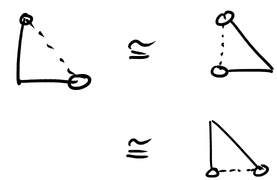
$$k\Delta = \{x\in\mathbb{R}^{d+1} \mid \sum x_i = k,\ x_i \geq 0\}$$

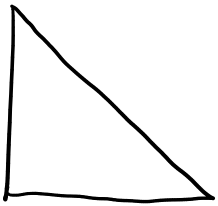
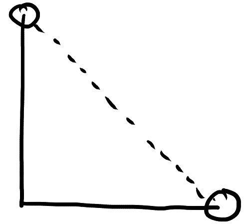
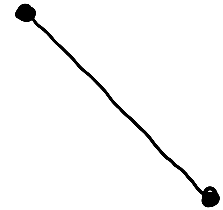
$$= \mathrm{conv}(ke_1,\dots,ke_{d+1})$$

$$\cong \{x\in\mathbb{R}^d \mid \sum x_i \leq k,\ x_i \geq 0\}$$

$$= \mathrm{conv}(0, ke_1,\dots,ke_d)$$

$L(k\Delta^d) = ?$

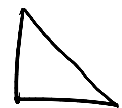


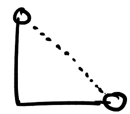
 =  + 

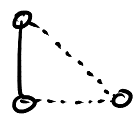
$k\Delta^2$ $\quad$ $k\Delta^2_1$ $\quad$ $k\Delta^1$

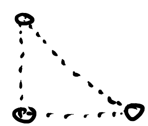
$$\Delta^d_i = \{ x \in \mathbb{R}^{d+1} \mid \sum x_i = 1, \; x_1 > 0, \dots, x_i > 0, \; x_{i+1} \geq 0, \dots, x_d \geq 0 \}$$

▷ d-simplex with i open faces!

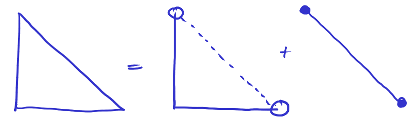
Δ^2_0 

Δ^2_1 

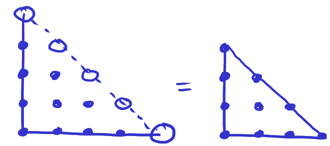
Δ^2_2 

Δ^2_3 

$$L(k\Delta^d) = L(k\Delta^d_1) + L(k\Delta^{d-1})$$



$$= L((k-1)\Delta^d) + L(k\Delta^{d-1})$$



$$L(0\Delta^d) = 1 \qquad L(k\Delta^0) = 1$$

$k \downarrow$ / $d \rightarrow$				
1	1	1	1	1
1	2	3	4	5
1	3	6	10	
1	4	10		

$$L(k\Delta^d) = \binom{d+k}{d}$$

$$L(k\Delta^d_i) = \binom{d+k-i}{d}$$

$$L(k\Delta^d_{d+1}) = \binom{k-1}{d}$$

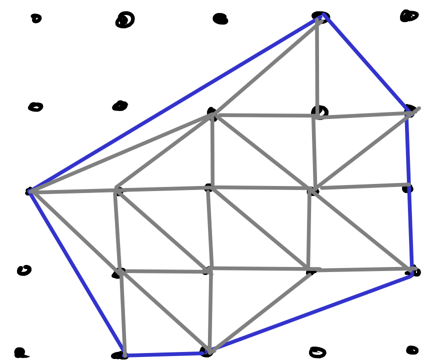
$$\binom{n}{d} = \frac{n^{\underline{d}}}{d!} = \frac{n(n-1)\cdot\ldots\cdot(n-d+1)}{d!}$$

This is a polynomial in n!

$L(k\Delta^d)$ is a polynomial in k of degree d!

$\to$ Does this work for arbitrary lattice polytopes?
$V(P) \subset \mathbb{Z}^d$

$\to$ It should, we can triangulate!



A simplex Δ is *unimodular* if it is a lattice transformation of the standard simplex Δ^d. So:

Lemma: $\Delta = \operatorname{conv}(v_0, v_1, \dots, v_d)$, with $v_0, v_1, \dots, v_d$ affinely indep.

Δ is unimodular $\iff \det(v_1 - v_0, \dots, v_d - v_0) = \pm 1$.

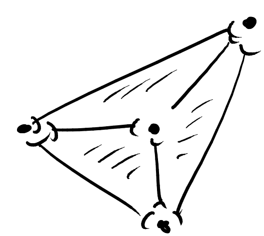
▷ Δ unimodular d-simplex $\Rightarrow L(k\Delta) = \binom{k+d}{d}$

▷ $\mathring{\Delta}$ *open* unimodular d-simplex $\Rightarrow L(k\mathring{\Delta}) = \binom{k-1}{d}$

▷ T a *unimodular simplicial complex* of dimension $\leq d$

$$\Rightarrow L(kT) = \sum_{i=0}^{d} f_i \binom{k-1}{i}$$

where $f_i = \#$ i-simplices in T



Ehrhart's Theorem

If P is a lattice polytope, then $L(kP)$ is a polynomial of degree dim P.

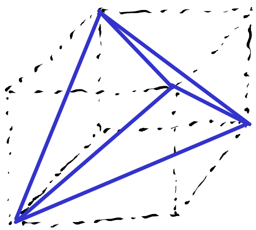
- If T is a unimodular triangulation of P, then
  $$L(kP) = L(kT) = \sum_{i=0}^{d} f_i \binom{k-1}{i}. \leftarrow \text{polynomial of degree } d!$$
- This is a proof if T has a unimodular triangulation.
- Not every polytope has a unimodular triangulation.
- But: theorem holds in general! (Even if there is no unimodular triangulation.)
- Define $L_P(k) := L(kP)$ for $k \in \mathbb{Z}_{>0}$.

  In general: $L_P(x) \neq L(xP)$!

  Ehrhart polynomial

  lattice point count.

Existence of Unimodular Triangulations

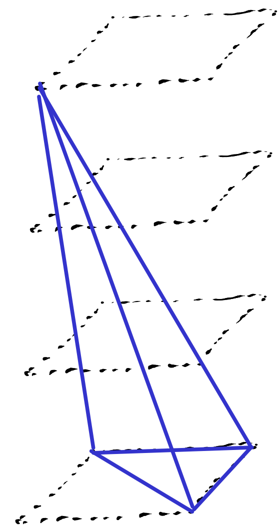


- ▷ Simplex without lattice points except vertices.
- ▷ $\Delta = \mathrm{conv}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$
- ▷ $\det\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix} = 1+0+1-0-0-0 = \underline{\underline{2}}$
- ▷ Not unimodular!

▷ Reeve's tetrahedra

- without lattice points except vertices
- with arbitrarily large volume!

$\det\begin{pmatrix}1&0&1\\0&1&1\\0&0&k\end{pmatrix} = k$



▷ Don't exist in dim 2!

$\Rightarrow$ all triangulations into "empty" simplices <u>are</u> unimodular in dimension 2!

$$\binom{k+d}{d} = \frac{(k+d)\cdot(k+d-1)\cdot\ldots\cdot(k+1)}{d!} \quad \text{is a polynomial in } k$$

$\Longrightarrow$ we can evaluate this at <u>negative</u> integers.

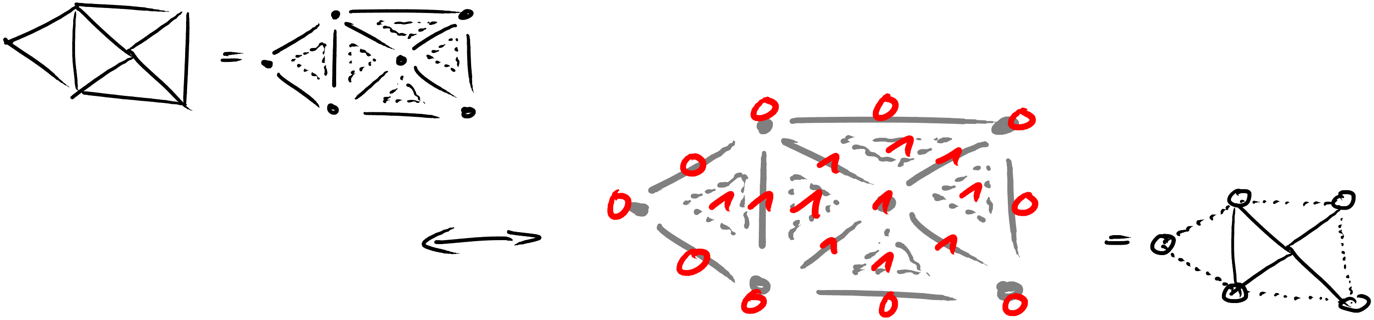
$$L_{\Delta^d}(-k) = \binom{-k+d}{d} = \frac{(-k+d)\cdot(-k+d-1)\cdot\ldots\cdot(-k+1)}{d!}$$

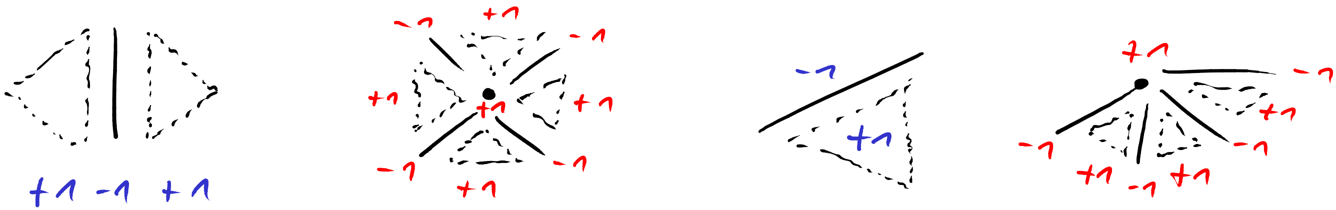
$$= (-1)^d \cdot \frac{(k-d)\cdot(k-d+1)\cdot\ldots\cdot(k-1)}{d!}$$

$$= (-1)^d \cdot \frac{(k-1)\cdot(k-2)\cdot\ldots\cdot(k-d)}{d!}$$

$$= (-1)^d \binom{k-1}{d} = (-1)^d \, L_{\Delta^d_{d+1}}(k)$$

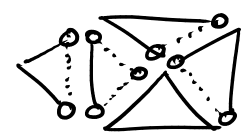
Δ^d_0 $\qquad\qquad$ $\Delta^d_{d+1} = \operatorname{relint} \Delta^d$

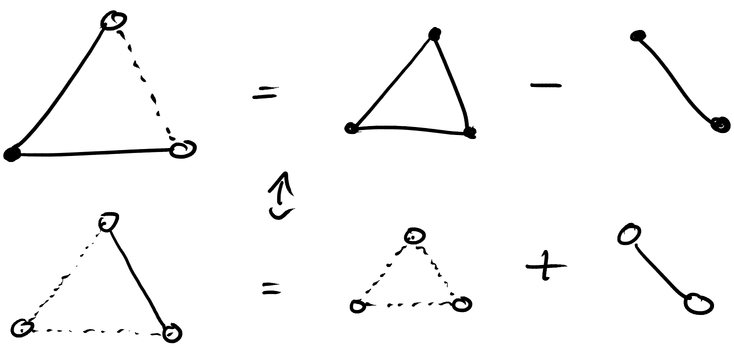




▷ This always works! (Requires work.)

▷ Can we use only simplices of top dimension, in order to avoid signs?

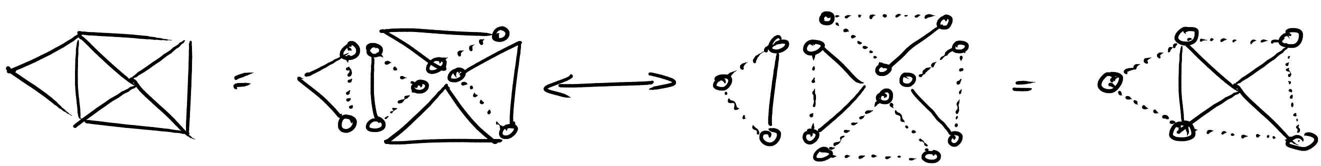




Lemma:

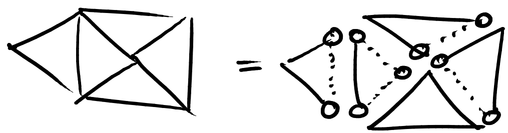
$$L_{\Delta_i^d}(-k) = (-1)^d L_{\Delta_{d+1-i}^d}(k)$$

$$\binom{-k+d-i}{d} = \frac{(-k+d-i)^{\underline{d}}}{d!} = (-1)^d \frac{(k-1+i)^{\underline{d}}}{d!} = (-1)^d \binom{k+d-(d+1-i)}{d}$$



Does this always work?

Shellings



$$L(kT) = L(k\Delta_0^2) + 3L(k\Delta_1^2) + L(k\Delta_2^2).$$

A simplicial complex is <u>pure</u>, if all maximal faces have the same dimension.

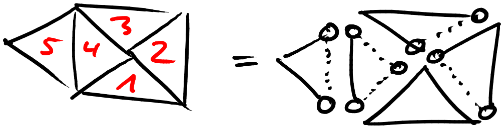
A <u>shelling</u> of a pure simplicial complex K of dimension d is a total order on the d-dim faces of K

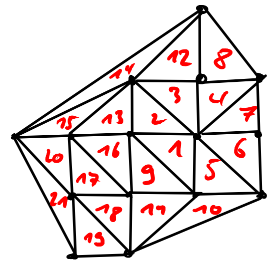
$$\sigma_1, \dots, \sigma_N$$

such that for every $j \geq 2$

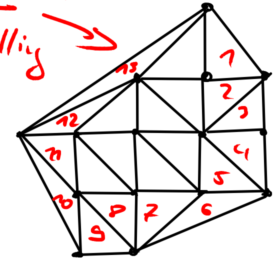
$$\left(\bigcup_{i=1}^{j-1} \sigma_i \right) \cap \sigma_j \quad \text{is pure } (d-1) \text{ dimensional.}$$

Examples

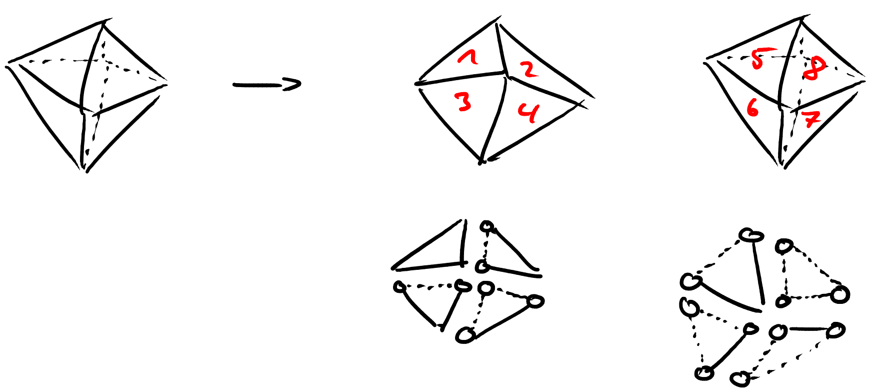


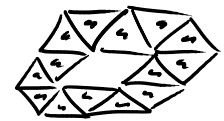


not a shelling

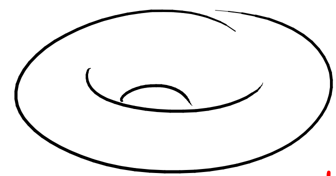


back face front face





not shellable!



not shellable!

Let $\sigma_1, \dots, \sigma_n$ be a shelling of simplicial comp. T

We say σ_j is of type ℓ for $0 \leq \ell \leq \dim T + 1$, if

$$\left(\bigcup_{i=1}^{j-1} \sigma_i \right) \cap \sigma_j$$ consists of ℓ d-simplices, i.e.,

$$\sigma_j \setminus \left(\bigcup_{i=1}^{j-1} \sigma_i \right)$$ has ℓ open faces.

Let h_ℓ = # type ℓ simplices in a shelling of T.

<u>Theorem</u>: T unimodular simplicial complex.

$$L(kT) = \sum_{i=0}^{d+1} h_i \binom{k+d-i}{d}$$

In particular: $0 \leq h_i \in \mathbb{Z}$.

Theorem: T unimodular simplicial complex.

$$L(kT) = \sum_{i=0}^{d+1} h_i \binom{k+d-i}{d}$$

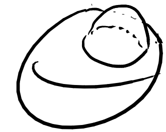
Observation If T is a ball, then there are no no type $d+1$ simplices!

Stanley's non-negativity theorem

If P is a lattice polytope, then

$$L_P(k) = \sum_{i=0}^{d} h_i \binom{k+d-i}{d}$$

where $\boxed{0 \leq h_i \in \mathbb{Z}}$ for all i.



Theorem: A d-dim shellable complex is homotopy equivalent to a wedge of d-spheres.

Theorem of Bruggesser and Mani: Boundary complexes of (simplicial) polytopes are shellable.

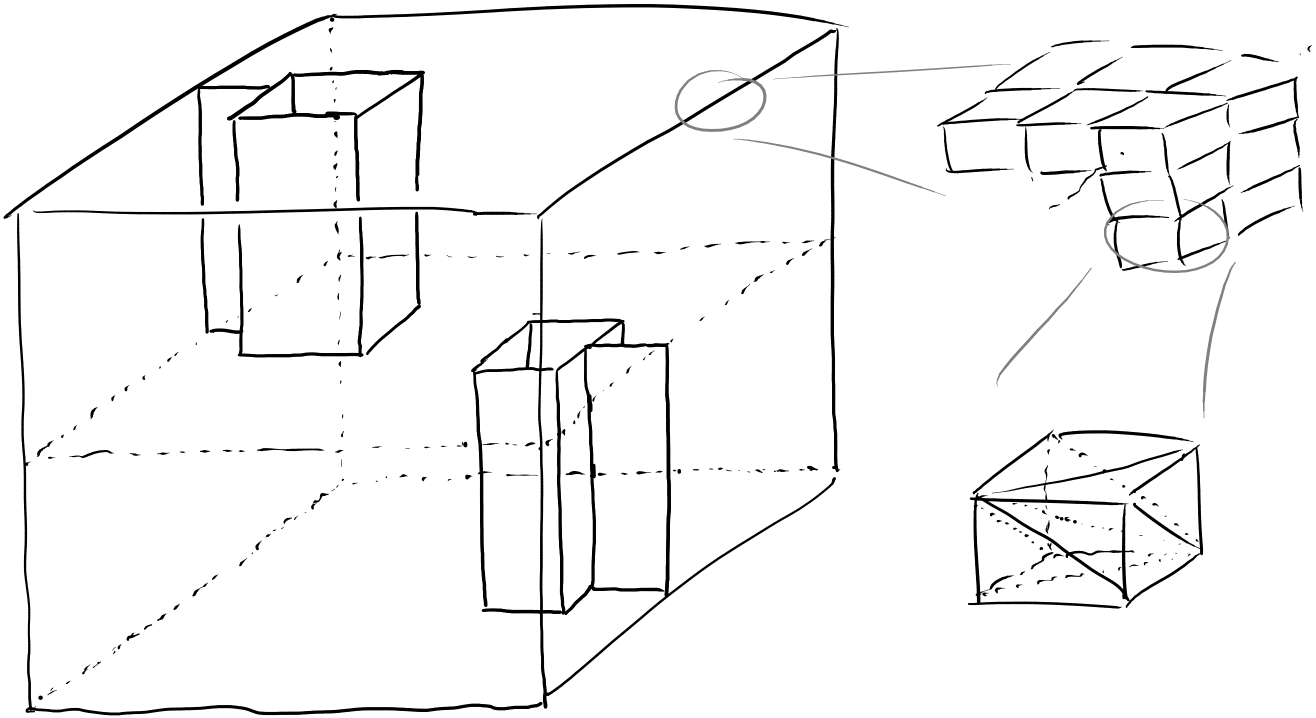
Poincaré-Conjecture / Perelman's Theorem

Every simply-connected, closed 3-manifold is homeomorphic to the 3-sphere.

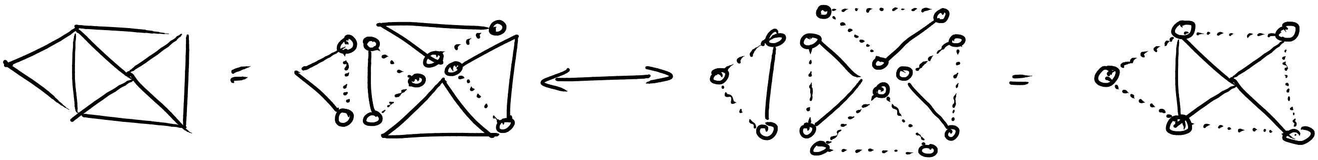
▷ Shellings were an attempt at a combinatorial characterization of spheres. But:

Theorem: There exist simplicial balls that are not shellable.

Bing's House with 2 Rooms



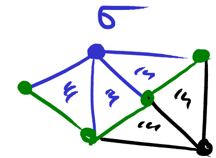
▷ a 3-dim **ball** that is **not shellable**



▷ σ a face of T

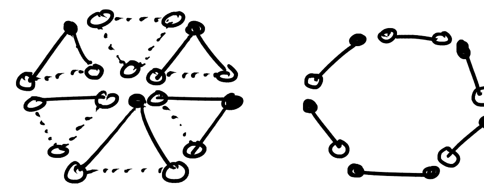
$\mathrm{star}(\sigma) = \{\sigma' \in T : \exists \hat{\sigma} : \sigma' \text{ and } \sigma \text{ are faces of } \hat{\sigma}\}$

$\mathrm{link}(\sigma) = \{\sigma' \in \mathrm{star}(\sigma) : \sigma \text{ is not a face of } \sigma'\}$

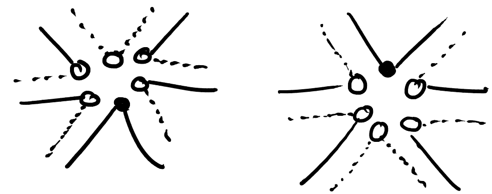


▷ a shelling of T induces a shelling of $\mathrm{star}(\sigma)$ and $\mathrm{link}(\sigma)$

▷ for every vertex in the <u>interior</u> of T $\mathrm{link}(v)$ is a <u>sphere</u>. this sphere has 1 type 0 and 1 type d simplex



▷ for every vertex on the <u>boundary</u> of T $\mathrm{link}(v)$ is a <u>ball</u>. this ball has 1 type 0 and 0 type d simplices



Theorem If T is a pure d-dimensional unimodular simplicial complex, such that

- link(σ) is a sphere for all σ in the "interior"
- link(σ) is a ball for all σ on the "boundary"

then $L_T(-k) = (-1)^d L_{\operatorname{relint} T}(k)$.

reciprocity recovers boundary

Ehrhart-Macdonald Reciprocity Theorem

If P is a lattice polytope, then

$$L_P(-k) = (-1)^{\dim P} L_{\operatorname{relint} P}(k).$$

- Works also if P is not a lattice polytope! (In which case $L_P(k)$ is a quasi-polynomial.)

Ehrhart Theory

$$L_P(k) = \#(kP \cap \mathbb{Z}^d)$$

- polynomiality (Ehrhart's Theorem)
- interpretation at negative integers (Ehrhart-Macdonald Reciprocity)
- bounds on coefficients (Non-negativity Thm)