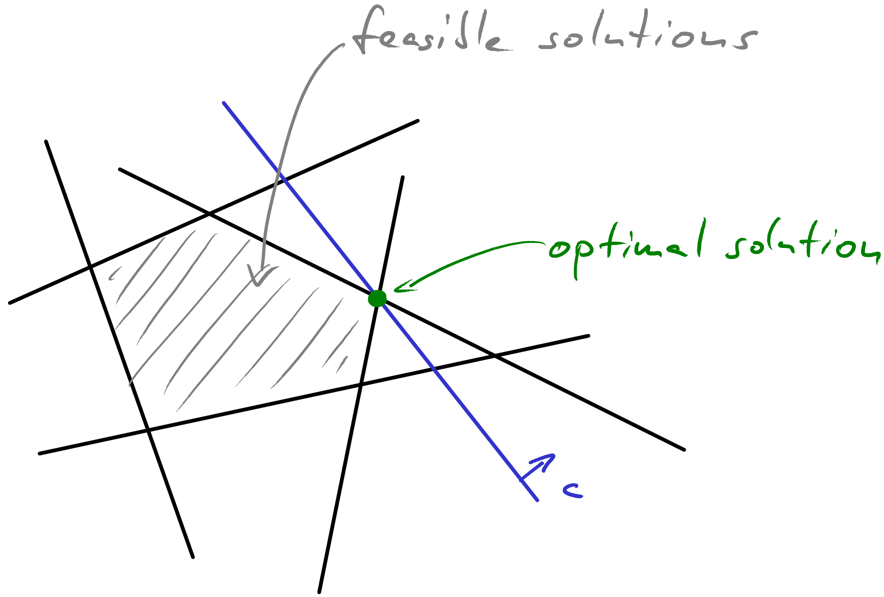
Linear Programs

$$\max \langle c, x\rangle$$
$$\text{subject to } Ax \le b$$



Instead of finding some solution to a system of linear inequalities we want to find a solution that maximizes a linear functional.

- extremely versatile
- operations research
  → whole industries are optimized in this way!
- many combinatorial problems can be modeled as linear programs.

Example: Max Flow Problem

Let $G=(V,E)$ be a directed graph.

Let $\delta^+(v) = \{e \in E \mid e=(w,v)\}$ and $\delta^-(v) = \{e\in E \mid e=(v,w)\}$

For any function $f: E \to \mathbb{R}$, the balance at $v \in V$ is

$$\text{balance}_f(v) = \sum_{e\in\delta^+(v)} f(v) - \sum_{e\in\delta^-(v)} f(v)$$

A sink is a vertex $v\in V$ with $\text{balance}_f(v) > 0$.

A source is a vertex $v \in V$ with $\text{balance}_f(v) < 0$.

Let $r,s \in V$. An <u>r-s-flow</u> is a function $f: E \to \mathbb{R}^+$ such that r is its only source and s is its only sink.

The value of an r-s-flow is $val(f) = \text{balance}_f(s) = -\text{balance}_f(r)$

f is subject to a capacity function $c: E \to \mathbb{R}^+$ if

$$f(e) \le c(e) \quad \text{for all } e \in E.$$

Question: What is the maximal value of an r-s-flow?

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Model this as a linear program

▷ view $f : E \to \mathbb{R}^+$ as a vector in $\mathbb{R}^E$.

▷ $\text{balance}_f(v) = \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e$ is linear in f !

▷ $\text{val}(f) = \text{balance}(s)$ is linear in f !

The Max Flow Problem gives rise to the linear program

$$
\begin{aligned}
& \max \quad \text{balance}(s) \\
\text{subject to:} \quad & f \geq 0 \\
& f \leq c \\
\forall v \in V, v \neq r, s: \quad & \text{balance}_f(v) = 0
\end{aligned}
$$

Linear Programming Duality

Primal:

$$\max \langle c, x \rangle.$$
$$\text{sub. to } \quad Ax \le b$$

Dual:

$$\min \langle b, y \rangle$$
$$\text{sub. to } \quad yA = c$$
$$y \ge 0$$

Theorem: If both $Ax \le b$ and $yA = c$, $y \ge 0$ have a solution, then

$$\max \{ \langle c, x \rangle \mid Ax \le b \} = \min \{ \langle b, y \rangle \mid yA = c, y \ge 0 \}.$$

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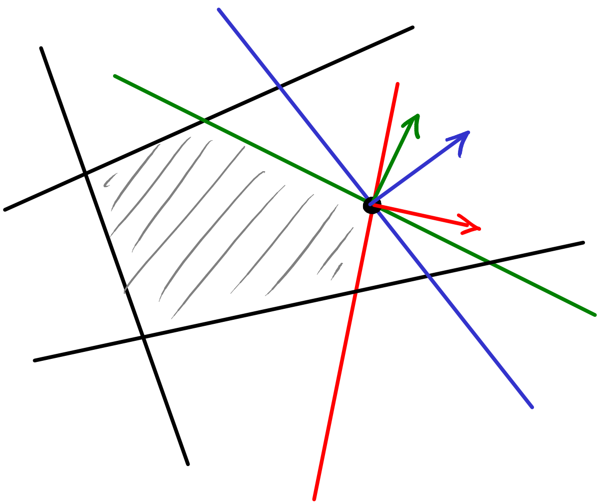
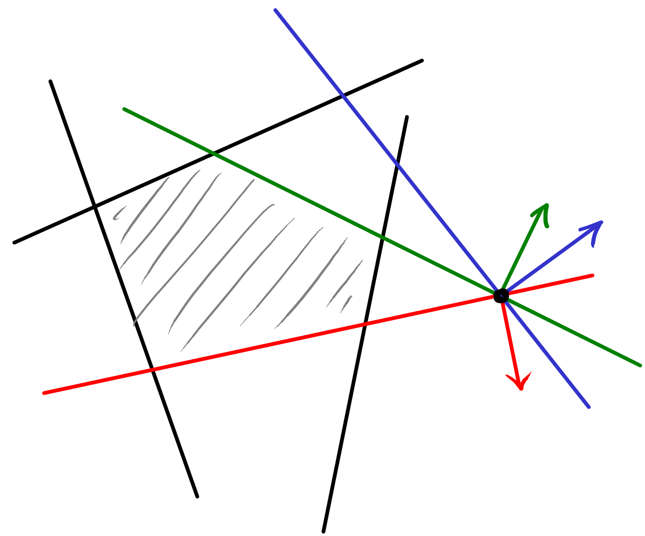




Theorem: If both $Ax \le b$ and $yA = c$, $y \ge 0$ have a solution, then

$$\max\{\langle c, x\rangle \mid Ax \le b\} = \min\{\langle b, y\rangle \mid yA = c, y \ge 0\}.$$

Proof:

1) Let x and y be solutions. Then

$$cx = yAx \le yb$$

$$\Rightarrow \quad \max \le \min$$

(in particular, both max and min are finite)

This is saying that

if $\lambda_i \ge 0$ then $\overline{H}_{\Sigma\lambda_i a_i, \Sigma\lambda_i b_i} \supset P$.

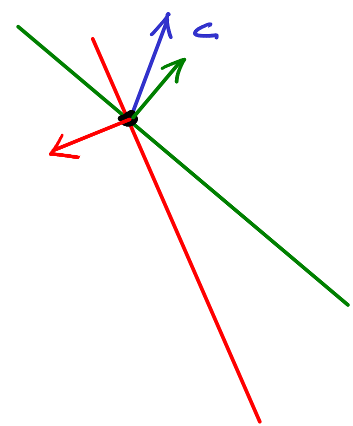
Theorem: If both $Ax \le b$ and $yA = c$, $y \ge 0$ have a solution, then

$$\max\{\langle c,x\rangle \mid Ax \le b\} = \min\{\langle b,y\rangle \mid yA = c, y \ge 0\}.$$

Proof: 2) $\max \ge \min$? To show: $\exists x, y : Ax \le b, y \ge 0, yA = c$
$yb = cx$

Intuition

The optimal solution x_0 is attained at a vertex.
Let B be the set of hyperplanes that contain the vertex x_0.
Then $A_B x_0 = b_B$ where A_B, b_B are subsets of rows of A, b.



Because x_0 is optimal, there exist
$y_i \ge 0$ s.t. $c = yA$ and $\mathrm{supp}(y) \subset B$.
Then

$$yb = y_B b_B = y_B A_B x_0 = yAx_0 = cx.$$

Theorem: If both $Ax \le b$ and $yA = c,\ y \ge 0$ have a solution, then

$$\max\{\langle c,x\rangle \mid Ax \le b\} = \min\{\langle b,y\rangle \mid yA = c,\ y \ge 0\}.$$

Proof: 2) $\max \ge \min$?

Want to show:

There exist x,y s.t.

$Ax \le b,\ yA = c,\ y \ge 0,\ yb \le cx$

$\iff$

There exist x,y s.t.

$$Ax \le b$$
$$A^T y^T \le c^T$$
$$-A^T y^T \le -c^T$$
$$b^T y^T - cx \le 0$$
$$-y \le 0$$

$\iff$

$$\begin{array}{cc} \begin{array}{c} m \\ n \\ n \\ 1 \\ m \end{array} & \left(\begin{array}{cc} A & 0 \\ 0 & A^T \\ 0 & -A^T \\ -c & b^T \\ 0 & -I \end{array}\right) \\ & \begin{array}{cc} n & m \end{array} \end{array} \begin{pmatrix} x \\ y^T \end{pmatrix} \le \underbrace{\begin{pmatrix} b \\ c^T \\ -c^T \\ 0 \\ 0 \end{pmatrix}}_{1}$$

Farkas Lemma

Either there exists a solution for $Ax \le b$
or there exists $y \ge 0,\ yA = 0,\ yb < 0$.

There exists a solution for $Ax \le b$
iff for all y: $y \ge 0,\ yA = 0 \Rightarrow yb \ge 0$.

$A \in \mathbb{R}^{m \times n}$ ← cols

↖ rows

$$
\begin{array}{cccccc}
\begin{array}{c} m \\ n \\ n \\ 1 \\ m \end{array} &
\begin{pmatrix} A & 0 \\ 0 & A^T \\ 0 & -A^T \\ -c & b^T \\ 0 & -I \end{pmatrix} &
\begin{pmatrix} x \\ y^T \end{pmatrix} & \le &
\begin{pmatrix} b \\ c^T \\ -c^T \\ 0 \\ 0 \end{pmatrix} \\
& \begin{array}{cc} n & m \end{array} & & & 1
\end{array}
$$

$$\Longleftrightarrow$$

Farkas Lemma

<u>Either</u> there exists a solution for $Ax \le b$
<u>or</u> there exists $y \ge 0$, $yA = 0$, $yb < 0$.

There exists a solution for $Ax \le b$
<u>iff</u> for all y : $y \ge 0$, $yA = 0 \Rightarrow yb \ge 0$.

$$
\begin{array}{l}
\text{For all } \overset{m}{\alpha}, \overset{n}{\beta}, \overset{n}{\gamma}, \overset{1}{\lambda}, \overset{m}{\mu} \ge 0 : \\
\text{if} \quad \alpha A - \lambda c = 0 \quad n \\
\qquad \beta A^T - \gamma A^T + \lambda b^T - \mu = 0 \quad m \\
\text{then} \quad \alpha b + \beta c^T - \gamma c^T \ge 0. \quad 1
\end{array}
$$

For all $\overset{m}{\alpha}, \overset{n}{\beta}, \overset{n}{\gamma}, \overset{1}{\lambda}, \overset{m}{\mu} \geq 0$:

if

$$\alpha A - \lambda c = 0 \quad n$$

$$\beta A^T - \gamma A^T + \lambda b^T - \mu = 0 \quad m \qquad \Longleftrightarrow \qquad \beta A^T - \gamma A^T + \lambda b^T \geq 0$$

then

$$\alpha b + \beta c^T - \gamma c^T \geq 0. \quad 1$$

Case 1: $\lambda > 0$.

$$\alpha b = \lambda^{-1} \lambda\, b^T \alpha^T \geq \lambda^{-1} (\gamma - \beta) A^T \alpha^T = \lambda^{-1} (\gamma - \beta) \lambda c^T$$

$$= (\gamma - \beta) c^T \qquad \checkmark$$

Case 2: $\lambda = 0$.

Let x_0, y_0 be solution of $Ax_0 \leq b$ and $y_0 \geq 0$, $yA = c$.

$$\alpha b \geq \alpha A x_0 = \lambda c x_0 = 0 \geq (\gamma - \beta) A^T y_0^T = (\gamma - \beta) c^T \qquad \checkmark$$

$$(\gamma - \beta) A^T \leq 0 \ \wedge \ y_0^T \geq 0 \ \Rightarrow \ (\gamma - \beta) A^T y_0^T \leq 0. \qquad \square$$

Complementary Slackness

Thm: Let x^*, y^* be feasible solutions to the dual problems

$$\max\{cx \mid Ax \leq b\} \quad \text{and} \quad \min\{yb \mid y \geq 0,\ Ay = c\}$$

Then the following are equivalent

1) x^* and y^* are optimal
2) $cx^* = y^*b$
3) $y_i^* > 0 \Rightarrow \langle a_i, x^* \rangle = b.$

if hyperplane i participates in the positive combination of c, then the hyperplane contains the optimum.

Proof: 1) $\Leftrightarrow$ 2) is the duality theorem

$$3) \Leftrightarrow \sum_i y_i^*(b - \langle a_i, x^* \rangle) = 0$$

$$\Leftrightarrow y^*(b - Ax^*)$$

$$\Leftrightarrow y^*Ax^* = y^*b^*$$

$$\Leftrightarrow cx^* = y^*b^* \quad \square$$

Note:

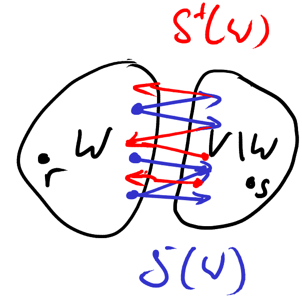
Either there exists an optimal solution with $\langle a_i, x \rangle < b_i$

or there exists an optimal solution with $y_i > 0$.

Duality and the Max Flow Problem

A <u>cut</u> in a graph is an ordered partition $W \dot\cup V \setminus W = V$.

An <u>r-s-cut</u> is a cut W with $r \in W$ and $s \notin W$.



$$\delta^-(W) = \{ e = (v_1, v_2) \in E \mid v_1 \in W,\ v_2 \notin W \}$$

$$\delta^+(W) = \{ e = (v_1, v_2) \in E \mid v_1 \notin W,\ v_2 \in W \}$$

The <u>capacity</u> of an r-s-cut W is

$$\sum_{e \in \delta^-(W)} c(e).$$

<u>Observation</u>:

Maximal value of an r-s-flow

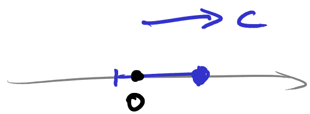
$\leq$ Minimal capacity of an r-s-cut.

Preparation

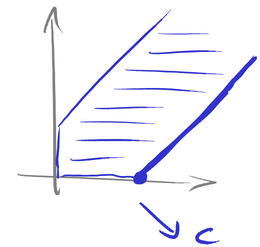
Every LP of the form $\begin{pmatrix} \max & cx \\ & Ax \leq b \end{pmatrix}$ is equivalent to an LP of the form $\begin{pmatrix} \max & cx \\ & Ax \leq b \\ & x \geq 0 \end{pmatrix}$.

Proof:

(1) $\max\ cx$, $Ax \leq b$



(2) $\max\ c(x' - x'')$, $A(x' - x'') \leq b$, $x', x'' \geq 0$



Convert solutions:

(1) $\longrightarrow$ (2)

$x_i' := x_i$ and $x_i'' := 0$ if $x_i \geq 0$

$x_i' := 0$ and $x_i'' := |x_i|$ if $x_i < 0$

▶ If x is feasible for (1) then x', x'' is feasible for (2).

(2) $\longrightarrow$ (1)

$x = x' - x''$

▶ If x', x'' is feasible for (2) then x is feasible for (1).

(1) $\max\ cx$
$Ax \le b$

(2) $\max\ c(x' - x'')$
$A(x' - x'') \le b$
$x', x'' \ge 0$

$(1) \longrightarrow (2)$

$x_i' := x_i$ and $x_i'' := 0$ if $x_i \ge 0$

$x_i' := 0$ and $x_i'' := |x_i|$ if $x_i < 0$

<u>Show:</u> x opt for (1) $\Rightarrow$ x', x'' opt for (2)

Suppose $\hat{x}', \hat{x}''$ are feasible and better:

$$c(\hat{x}' - \hat{x}'') > c(x' - x'')$$

$$\Rightarrow \quad c\hat{x} > cx$$

for $\hat{x} = \hat{x}' - \hat{x}''$. ↯

$(2) \longrightarrow (1)$

$$x = x' - x''$$

<u>Show:</u> x', x'' opt for (2) $\Rightarrow$ x opt for (1)

Suppose $\hat{x}$ is feasible and better: $c\hat{x} > cx$

Define $\hat{x}', \hat{x}''$ as above.

$$\Rightarrow \quad c(\hat{x}' - \hat{x}'') > c(x' - x'')$$ ↯

Apply Duality to Max Flow Problem:

$$\max \ \mathrm{balance}(s)$$

sub. to:

$$\begin{aligned} -f &\le 0 \\ f &\le c \\ \mathrm{balance}_f(v) &\le 0 \\ -\mathrm{balance}_f(v) &\le 0 \end{aligned} \left.\vphantom{\begin{aligned} a \\ b \end{aligned}}\right\} \begin{aligned} &\forall v \in V, \\ &v \neq r, s \end{aligned}$$

$$\min \ (\overbrace{\alpha, \beta}^{E}, \overbrace{\gamma, \delta}^{V-r,s}) \begin{pmatrix} 0 \\ c \\ 0 \\ 0 \end{pmatrix} = \beta c$$

sub to:

$$\alpha, \beta, \gamma, \delta \ge 0$$

$$(\alpha, \beta, \gamma, \delta) \underbrace{\begin{pmatrix} -I \\ I \\ Inc \\ -Inc \end{pmatrix}}_{\text{edges}} = \begin{pmatrix} a_{se} \end{pmatrix}$$

where

$$Inc = (a_{ve})_{\substack{v \in V-r,s \\ e \in E}}$$

$$a_{ve} = \begin{cases} 1 & \text{if } e = (w, v) \\ -1 & \text{if } e = (v, w) \\ 0 & \text{otherwise} \end{cases}$$

$$a_{se} = -\alpha_e + \beta_e + \sum_{v \neq r,s} \gamma_v a_{ve} - \sum_{v \neq r,s} \delta_v a_{ve}$$

$$a_{se} = -\alpha_e + \beta_e + \sum_{v \neq r,s} \gamma_v a_{ve} - \sum_{v \neq r,s} \delta_v a_{ve}$$

$$= -\alpha_e + \beta_e + \sum_{v \neq r,s} \underbrace{(\gamma_v - \delta_v)}_{z_v} a_{ve}$$

$z_v \in \mathbb{R}$

$v \in V - \{r,s\}$

$$a_{se} = -\alpha_e + \beta_e + \sum_{v \neq r,s} z_v a_{ve}$$

$$\Leftrightarrow \quad 0 = -\alpha_e + \beta_e + \sum_{v \in V} z_v a_{ve}$$

where $z_s = -1$ and $z_r = 0$

$$\Rightarrow \quad \alpha_e \leq \beta_e + \sum_{v \in V} z_v a_{ve} = \beta_e + z_{w_0} - z_{v_0}$$

where $e = (v_0, w_0)$

$$\Rightarrow \quad 0 \leq \beta_e + z_{w_0} - z_{v_0}$$

$z \in \mathbb{R}^V, \; y \in \mathbb{R}^E$

max balance(s)

sub. to:

$$-f \leq 0$$
$$f \leq c$$
$$\left.\begin{array}{r} \text{balance}_f(v) \leq 0 \\ -\text{balance}_f(v) \leq 0 \end{array}\right\} \forall v \in V,\ v \neq r, s$$

min yc

sub to:

$$0 \leq y_e + z_w - z_v \quad \text{for } e=(v,w)$$
$$0 \leq y_e$$
$$z_s = -1$$
$$z_r = 0$$

Let f^*, y^*, z^* be optimal solutions. Define $W = \{ v \mid z^*_v \geq 0 \}$.

Let $e=(v_1,v_2) \in \delta^-(W)$. $\Rightarrow z_{v_1} > z_{v_2} \Rightarrow$

$$y_e \geq z_{v_1} - z_{v_2} > 0 \overset{\text{c.s.}}{\Longrightarrow} f^*_e = c.$$

Let $e=(v_1,v_2) \in \delta^+(W) \Rightarrow z_{v_2} > z_{v_1} \Rightarrow$

$$y_e + z_{v_2} - z_{v_1} \geq z_{v_2} - z_{v_1} > 0 \overset{\text{c.s}}{\Longrightarrow} f^*_e = 0.$$

$$\text{balance}_{f^*}(r) = \sum_{e\in\delta^+(r)} f^*(e) - \sum_{e\in\delta^-(r)} f^*(e)$$

$$= \sum_{v\in W}\left(\sum_{e\in\delta^+(v)} f^*(e) - \sum_{e\in\delta^-(v)} f^*(e)\right)$$

$$= \sum_{e\in\delta^+(W)} f^*(e) - \sum_{e\in\delta^-(W)} f^*(e) = 0 - \sum_{e\in\delta^-(W)} c_e$$

$\text{val}_f = \sum_{e\in\delta^-(W)} c_e$. = capacity of the r-s-cut W !

<u>Theorem</u> (Max Flow - Min Cut)

The value of a maximal r-s-flow equals the capacity of a minimal r-s-cut.