Fourier-Motzkin Elimination

$\Downarrow$

Farkas Lemma

$\Downarrow$

LP-Duality $\Longleftrightarrow$ Minimax-Theorem

Sperner's Lemma

$\Downarrow$

Brouwer's Fixpoint Thm

$\Downarrow$

Minimax-Theorem

LP-Duality $\Downarrow$ Complementary Slackness

$\Downarrow$

Max-Flow Min-Cut

LP-Duality $\Downarrow$ Simplex Algorithm

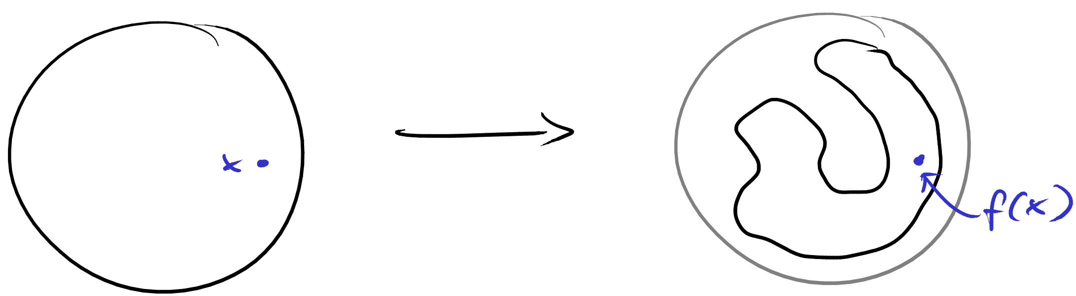
Brouwer's Fixpoint Theorem

Let $X \subset \mathbb{R}^n$ be compact and convex. (e.g. a ball)

Then any continuous map

$$f : X \longrightarrow X$$

has a fixpoint: there exists $x \in X$ s.t. $f(x) = x$.



Brouwer's Fixpoint Thm $\Rightarrow$ Minimax Theorem

Minimax Theorem $\min_y \max_x K(x,y) = \max_x \min_y K(x,y)$

$$\Leftrightarrow \exists x^*, y^* : \forall x, y : K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y)$$

Proof: $f : \Delta^{S_{Max}} \times \Delta^{S_{Min}} \to \Delta^{S_{Max}} \times \Delta^{S_{Min}}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \arg\max_{x'} K(x', y) \\ \arg\min_{y'} K(x, y') \end{pmatrix}$$

▷ f is continuous (!) $\Rightarrow$ there is a fixpoint x^*, y^*.

▷ $x^* = \arg\max_{x'} K(x', y^*) \Rightarrow K(x', y^*) \leq K(x^*, y^*) \quad \forall x'$

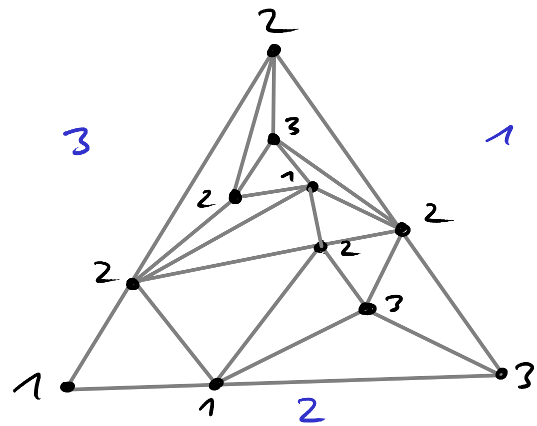
▷ $y^* = \arg\min_{y'} K(x^*, y') \Rightarrow K(x^*, y^*) \leq K(x^*, y') \quad \forall y'$

Note: here we assumed that argmax and argmin are unique. $\square$

Sperner's Lemma

Let Δ^n be the n-simplex and K a triangulation.

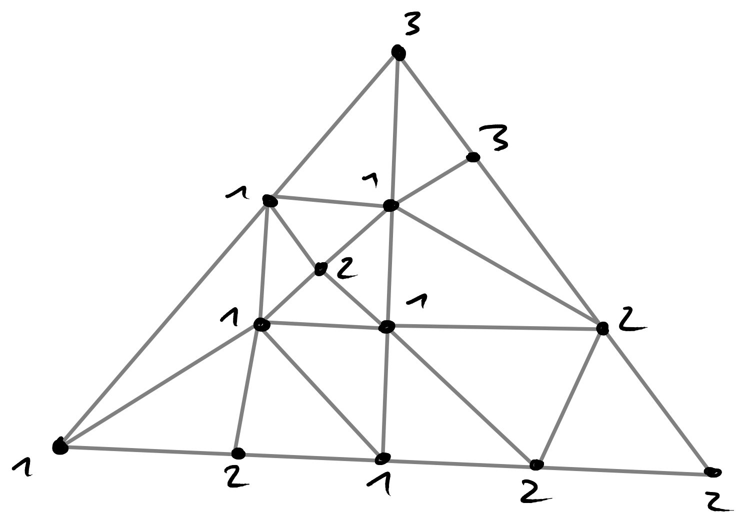
Let c be a coloring of the vertices with $n+1$ colors.



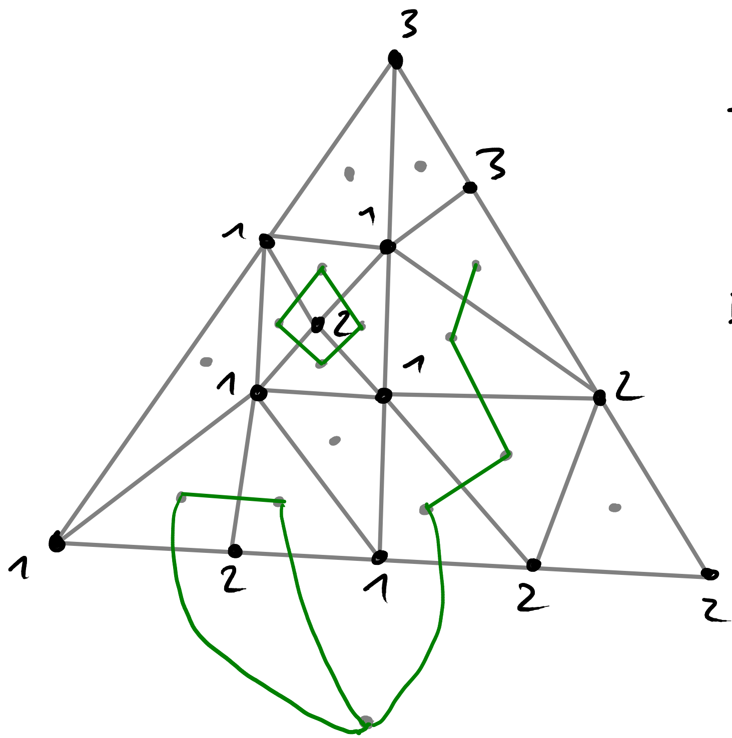
c is a <u>Sperner labeling</u> if side i does <u>not</u> contain color i.

Sperner's Lemma

In any Sperner labeling of K there is a simplex that is labeled with <u>all</u> colors $1, \dots, n+1$.



Proof:



▷ Construct a graph
- with a vertex for each simplex of K
- with an edge between any two simplexes that share an edge with labels 1, 2

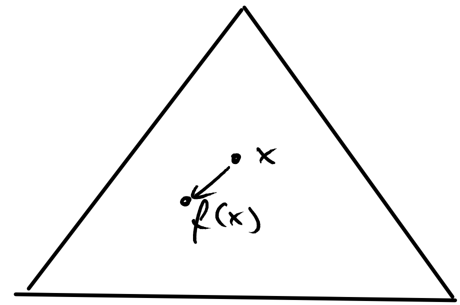
▷ The "outside" vertex has odd degree.

▷ The "inside" vertices have odd degree if and only if they are labeled 1, 2, 3.

▷ Every graph has an even number of vertices of odd degree.

□

Sperner ⇒ Brouwer (rough idea)



- If f has no fixpoint, consider

$$g : x \longmapsto f(x) - x$$

- Construct a sequence of finer and finer triangulations K_i.
- Color vertex v according to towards which vertex $g(v)$ points.
- Convergence argument:

fixpoint $\Longleftrightarrow$ fully labeled simplices.

□